Legendre differential equation and Legendre functions:

Legendre equation

The differential equation

$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$ ------- (1)

$\frac{d}{dx}\{(1 - x^2)\frac{dy}{dx}\} + n(n+1)y = 0$ is called Legendre's equation.

Solution:

$y = x^k(a_0 + a_1x^{-1} + a_2x^{-2} + a_3x^{-3} + a_4x^{-4} ----+ a_rx^{-r} ----)$

$y = \sum_{r=0}^{\infty} a_r x^{k-r}$ ------- (2)

$\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r(k-r)x^{k-r-1}$

$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r(k-r)(k-r-1)x^{k-r-2}$

Substituting these values in equation (1)

$(1 - x^2)\sum_{r=0}^{\infty} a_r(k-r)(k-r-1)x^{k-r-2} - 2x\sum_{r=0}^{\infty} a_r(k-r)x^{k-r-1} + n(n+1)\sum_{r=0}^{\infty} a_rx^{k-r} = 0$

$\sum_{r=0}^{\infty} a_r[(k-r)(k-r-1)x^{k-r-2} - (k-r)(k-r-1)x^{k-r} - 2(k-r)x^{k-r} + n(n+1)x^{k-r}] = 0$

$\sum_{r=0}^{\infty} a_r[(k-r)(k-r-1)x^{k-r-2} + \{n(n+1) - (k-r)(k-r-1+2)\}x^{k-r}] = 0$

$\sum_{r=0}^{\infty} a_r[(k-r)(k-r-1)x^{k-r-2} + \{n(n+1) - (k-r)(k-r+1)\}x^{k-r}] = 0$ ------- (3)

This equation is an identity; therefore the coefficients of various powers of x must be equal to zero.Let us first put the coefficients of x^k equal to zero by substituting r = 0 (The highest power of x is k).

$\{n(n+1) - k(k+1)\}\ a_0 = 0$

Where a_0 being the coefficients of the first term of the series is not equal to zero, hence

$n(n+1) - k(k+1)=0$

$n^2 + n - k^2 - k = 0$

$(n^2 - k^2) + (n - k) = 0$

$(n - k)(n + k + 1) = 0$

K = n or k = -(n+1) ----------(4)

Again, equating the coefficients of x^{k-1} to zero by putting r= 1 in equation (3)

$[n(n+1) - (k-1)k]a_1 = 0$

$[n^2 + n - k^2 + k]a_1 = 0$

$[(n^2 - k^2) + (n + k)]a_1 = 0$

$(n + k)(n - k + 1)a_1 = 0$

As $(n + k)(n - k + 1) \neq 0;\ therefore\quad a_1 = 0$--------------(5)

To obtain a general relation between coefficients of series; Let us equate the coefficients of x^{k-r-2} in equation (3).

$a_r(k - r)(k - r - 1) + \{n(n + 1) - (k - r - 2)(k - r - 1)\}a_{r+2} = 0$

$$a_{r+2} = -\frac{(k-r)(k-r-1)}{n(n+1)-(k-r-2)(k-r-1)}\ a_r$$

But

$(k - r - 2)(k - r - 1) - n(n + 1)$

$(k - r)^2 - 2(k - r) - (k - r) + 2 - n^2 - n$

$(k - r)^2 - (k - r)(2 + 1) - (n^2 + n - 2)$

$(k - r)^2 - (k - r)(n + 2 - n + 1) - (n^2 + n - 2)$

$(k - r)^2 - (k - r)((n + 2) + (k - r)(n - 1) - (n - 1)(n + 2)$

$(k - r)\{(k - r) - (n + 2)\} + (n - 1)\{(k - r) - (n + 2)$

$\{(k - r) + (n - 1)\}\{(k - r) - (n + 2)\}$

$$a_{r+2} = \frac{(k-r)(k-r-1)}{\{(k-r)+(n-1)\}\{(k-r)-(n+2)\}}\ a_r \quad \text{--------(6)}$$

As $a_1 = 0$, therefore equation (6) implies that

$a_1 = a_3 = a_5 = a_7 = ------- = 0$

i.e. all the coefficients a's having odd suffixes are zero

Case (i): when k = n, we get from equation (6)

$$a_{r+2} = \frac{(n-r)(n-r-1)}{\{(n-r)+(n-1)\}\{(n-r)-(n+2)\}}\ a_r$$

$$a_{r+2} = \frac{(n-r)(n-r-1)}{(2n-r-1)(-r-2)}\ a_r$$

$$a_{r+2} = -\frac{(n-r)(n-r-1)}{(2n-r-1)(r+2)}\ a_r$$

Substituting r = 0,2,4 ---- , we get

For r = 0 $\quad a_2 = -\frac{n(n-1)}{(2n-1).2}\ a_0$

For r = 2 $\quad a_4 = -\frac{(n-2)(n-3)}{(2n-3).4}\ a_2 = \frac{n(n-1)(n-2)(n-3)}{(2n-1)(2n-3).2.4}\ a_0$

For r = 4 $\quad a_6 = -\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{(2n-1)(2n-3)(2n-5).2.4.6}\ a_0$

and so on

Also we have $a_1 = a_3 = a_5 = a_7 = ------- = 0$

Substituting values of various coefficients a's in equation (2), we get the series solution for k=n as

$$y = \sum_{r=0}^{\infty} a_r x^{n-r}$$

$$y = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + a_4x^{n-4} + a_5x^{n-5} + a_6x^{n-6} --------$$

$$y = a_0[x^n - \frac{n(n-1)}{(2n-1).2}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{(2n-1)(2n-3).2.4}x^{n-4} - \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{(2n-1)(2n-3)(2n-5).2.4.6}x^{n-6} ----]$$

Where a_0 is arbitrary constant and n is positive integer if

$$a_0 = \frac{1.3.5.7-----(2n-1)}{n!}$$

Then above solution is called the ***Legendre polynomial*** or ***Legendre function of first kind*** and is represented by $P_n(x)$

$$p_n(x) = \frac{1.3.5.7-----(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{(2n-1).2}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{(2n-1)(2n-3).2.4}x^{n-4} - \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{(2n-1)(2n-3)(2n-5).2.4.6}x^{n-6} ----\right]$$

This series is a terminating series and for different values of n we get Legendre polynomials.

Case (ii):

when k = -n-1, we get from equation (6)

$$a_{r+2} = \frac{(-n-1-r)(-n-1-r-1)}{\{-n-1-r+n-1)\}\{-n-1-r-n-2)\}}\ a_r$$

$$a_{r+2} = \frac{(n+r+1)(n+r+2)}{(r+2)(2n+r+3)}\ a_r$$

Substituting r = 0,2,4 ---- , we get

For r = 0 $\quad a_2 = \frac{(n+1)(n+2)}{(2n+3).2}\ a_0$

For r = 2 $\quad a_4 = \frac{(n+3)(n+4)}{(2n+5).4}\ a_2 = \frac{(n+1)(n+2)(n+3)(n+4)}{(2n+3)(2n+5).2.4}\ a_0$

For r = 4 $\quad a_6 = \frac{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)}{(2n+3)(2n+5)(2n+7).2.4.6}\ a_0$

and so on

Also we have $a_1 = a_3 = a_5 = a_7 = ------- = 0$

Substituting values of various coefficients a's in equation (2), we get the series solution for k= -n-1 as

$$y = \sum_{r=0}^{\infty} a_r x^{-n-1-r}$$

$$y = a_0x^{-n-1} + a_1x^{-n-2} + a_2x^{-n-3} + a_3x^{-n-4} + a_4x^{-n-5} + a_5x^{-n-6} + a_6x^{-n-7} --------$$

$$y = a_0\left[x^n + \frac{(n+1)(n+2)}{(2n+3).2}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{(2n+3)(2n+5).2.4}x^{-n-5} + \frac{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)}{(2n+3)(2n+5)(2n+7).2.4.6}x^{-n-7} --\right]$$

If $a_0 = \frac{n!}{1.3.5.7-----(2n-1)}$; the above solution is called ***Legendre polynomial*** or **Legendre function of second kind** and denoted by $Q_n(x)$

$$Q_n(x) = \frac{n!}{1.3.5.7-----(2n-1)}\left[x^n + \frac{(n+1)(n+2)}{(2n+3).2}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{(2n+3)(2n+5).2.4}x^{-n-5} + \frac{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)}{(2n+3)(2n+5)(2n+7).2.4.6}x^{-n-7} ---\right]$$

This is an infinite or non-terminating series, sense is a positive integer.

As $p_n(x)$ and $Q_n(x)$ are two independent solutions of Legendre equation: therefore the most general solution of Legendre equation may be expressed as

$y = A\ p_n(x) + B\ Q_n(x)$ Where A and B are arbitrary constants.