Formulas used

1. $(1-x)^{-1/2} = 1+\frac{1}{2}x + \frac{1}{2}.\frac{3}{4}x^2 + \frac{1}{2}.\frac{3}{4}.\frac{5}{6}x^3 -----+ \frac{1.3.5.7---(2n-3)}{2.4.6----(2n-2)}x^{n-1} + \frac{1.3.5.7---(2n-1)}{2.4.6----+2n}x^n +---$

2. $(1-x)^{-1} = 1+x+x^2+x^3 -----$

3. $(x+y)^n = n_{C_0}x^n + n_{C_1}x^{n-1}y + n_{C_2}x^{n-2}y^2 + n_{C_3}x^{n-3}y^3 +-------$

$$---+ n_{C_r}x^{n-r}y^r + ---- n_{C_n}y^n$$

4. $n_{C_r} = \frac{n!}{r!(n-r)!}$

5. 0! = 1

6. $\int u dv = uv - \int v du$

Generating Function of Legendre Polynomial

Theorem: To show that $P_n(x)$ is the coefficient of z^n in the expansion of $[1-2xz+z^2]^{-1/2}$ in ascending powers of z.

We have $[1-2xz+z^2]^{-1/2} = [1-z(2x-z)]^{-1/2}$

$= 1+\frac{1}{2}z(2x-z) + \frac{1}{2}.\frac{3}{4}z^2(2x-z)^2 + \frac{1}{2}.\frac{3}{4}.\frac{5}{6}z^3(2x-z)^3 -------$

$---+ \frac{1.3.5.7---(2n-3)}{2.4.6----(2n-2)}z^{n-1}(2x-z)^{n-1} + \frac{1.3.5.7---(2n-1)}{2.4.6----2n}z^n(2x-z)^n + \quad --- \quad ...(1)$

Now coefficient of z^n in $\frac{1.3.5.7---(2n-1)}{2.4.6----2n}z^n(2x-z)^n$ is clearly

$\frac{1.3.5.7---(2n-1)}{2.4.6----2n}(2x)^n = \frac{1.3.5.7---(2n-1)}{2^n(1.2.3---n)}2^n x^n = \frac{1.3.5.7---(2n-1)}{n!} x^n$ ----(i)

Also coefficient of z^n in $\frac{1.3.5.7---(2n-3)}{2.4.6----(2n-2)}z^{n-1}(2x-z)^{n-1}$ is

$\frac{1.3.5.7---(2n-3)}{2.4.6----(2n-2)}[-(n-1)(2x)^{n-2}] =- \frac{1.3.5.7---(2n-3)(2n-1)}{2^{n-1}[1.2.3---(n-1)]n}.\frac{n(n-1)}{(2n-1)}2^{n-2} x^{n-2}$

$\frac{2^{n-2}.}{2^{n-1}} = \frac{2^{n-2}.2.}{2^{n-1}2.} = \frac{2^{n-1}}{2^{n-1}2.} = \frac{1}{2.}$	$(n-1)_{C_1} = \frac{(n-1)!}{1!.(n-1-1)!} = \frac{(n-1)(n-2)(n-3)----}{(n-2)(n-3)-----}$=(n-1)

$=- \frac{1.3.5.7---(2n-1)}{n!}.\frac{n(n-1)}{(2n-1).2} x^{n-2}$------(ii)

Similarly, the coefficient of z^n in $\frac{1.3.5.7---(2n-5)}{2.4.6----(2n-4)} z^{n-2}(2x - z)^{n-2}$ is

$$(n - 2)_{C_2} = \frac{(n-2)!}{2!.(n-2-2)!} = \frac{(n-2)(n-3)(n-4)(n-5)(n-6)----}{2!(n-4)(n-5)(n-6)-----} = \frac{(n-2)(n-3)}{2!}$$

$$\frac{1.3.5.7---(2n-5)}{2.4.6----(2n-4)} [\frac{(n-2)(n-3)}{2!}(2x)^{n-4}] \frac{(2n-3)(2n-1)(n-1)n}{(2n-3)(2n-1)(n-1)n}$$

$$\frac{1.3.5.7---(2n-1)}{2^{n-2}(1.2.3----(n-2)(n-1)n} 2^{n-4} x^{n-4} \frac{n(n-1)(n-2)(n-3)}{2.(2n-3)(2n-1)}$$

$$\frac{2^{n-4}.}{2^{n-2}} = \frac{2^{n-4}.2^2.}{2^{n-2}2^2.} = \frac{2^{n-2}}{2^{n-2}2^2.} = \frac{1}{4.}$$

$$\frac{1.3.5.7---(2n-1)}{n!} \quad \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-3)(2n-1)} x^{n-4}$$ ----- (iii)

From (i),(ii),(iii), the coefficient of z^n in the expansion of (1) is

$$\frac{1.3.5.7---(2n-1)}{n!} [x^n - \frac{n(n-1)}{(2n-1).2} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-3)(2n-1)} x^{n-4} - --------]$$

Which is obviously $P_n(x)$

Thus we can say that in the expansion of $[1 - 2xz + z^2]^{-1/2}$, the coefficients of z,z²,z³----etc will be $P_1(x) + P_2(x) + P_3(x) +-------$. Hence we can write

$$[1 - 2xz + z^2]^{-1/2} = 1 + zP_1(x) + z^2P_2(x) + z^3P_3(x) +-------+ z^nP_n(x) +----$$

$$[1 - 2xz + z^2]^{-1/2} = \sum_{n=0}^{\infty} z^nP_n(x)$$ ---------(2)

Corollary1. To show that $P_n(1) = 1$

Substituting x=1 on either side of equation (2); we get

$$[1 - 2z + z^2]^{-1/2} = \sum_{n=0}^{\infty} z^nP_n(1)$$

$$\{(1 - z)^2\}^{-1/2} = 1 + zP_1(1) + z^2P_2(1) + z^3P_3(1) +-------+ z^nP_n(1) +----$$

$$(1 - z)^{-1} = 1 + zP_1(1) + z^2P_2(1) + z^3P_3(1) +-------+ z^nP_n(1) +----$$

$$1 + z + z^2 + z^3 +---+ z^n +---= 1 + zP_1(1) + z^2P_2(1) + z^3P_3(1) +-------+ z^nP_n(1) +----$$

Equating coefficients of z^non either side we get

$P_n(1) = 1$ -------- (3)

Corollary2. $P_n(-x) = (-1)^n P_n(x)$

We have $[1 - 2xz + z^2]^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x)$

Substituting -x of x in equation (2); we get

$[1 + 2xz + z^2]^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(-x)$ ------ (4)

Again substituting -z for z in equation (2); we get

$$[1 + 2xz + z^2]^{-1/2} = \sum_{n=0}^{\infty} (-z)^n P_n(x) = \sum_{n=0}^{\infty} (-1)^n (z)^n \ P_n(x) \quad ---- \ (5)$$

Comparing (4) and (5), we get

$$P_n(-x) = (-1)^n P_n(x)$$