Rodrigue's Formula for Legendre Polynomials

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n$$

Let $y = (x^2 - 1)^n$ ----- (1)

$\frac{dy}{dx} = n(x^2 - 1)^{n-1}.2x$ ---- (2)

Multiplying both sides of above equation (2) by $(x^2 - 1)$: we get

$$(x^2 - 1)\frac{dy}{dx} = n(x^2 - 1)^{n-1}.2x.(x^2 - 1)$$

$(x^2 - 1)\frac{dy}{dx} = n(x^2 - 1)^n.2x \Rightarrow (x^2 - 1)\frac{dy}{dx} = 2nxy$ ------ (3)

Leibniz's theorem

$$D^n(uv) = (D^n u)v + {}^nC_1(D^{n-1}u)(Dv) + {}^nC_2(D^{n-2}u)D^2(v) + ---$$
$$---- + {}^nC_r(D^{n-r}u)D^r(v) +-----+ {}^nC_n u D^n(v)$$

Differentiating this equation (n+1) times by Leibniz's theorem

$$(x^2 - 1)\frac{d^{n+2}y}{dx^{n+2}} + {}^{n+1}C_1\frac{d^{n+1}y}{dx^{n+1}}(2x) + {}^{n+1}C_2\frac{d^n y}{dx^n}(2)$$

$$= 2n[x\frac{d^{n+1}y}{dx^{n+1}} + {}^{n+1}C_1\frac{d^n y}{dx^n}]$$

$$(x^2 - 1)\frac{d^{n+2}y}{dx^{n+2}} + 2(n + 1)x\frac{d^{n+1}y}{dx^{n+1}} + n(n + 1)\frac{d^n y}{dx^n} = 2nx\frac{d^{n+1}y}{dx^{n+1}} + 2n(n + 1)\frac{d^n y}{dx^n}$$

$$(x^2 - 1)\frac{d^{n+2}y}{dx^{n+2}} + 2x(n + 1 - n)x\frac{d^{n+1}y}{dx^{n+1}} - n(n + 1)[2 - 1]\frac{d^n y}{dx^n} = 0$$

$(1 - x^2)\frac{d^{n+2}y}{dx^{n+2}} - 2x\frac{d^{n+1}y}{dx^{n+1}} + n(n + 1)\frac{d^n y}{dx^n} = 0$ ------ (4)

Substituting $\frac{d^n y}{dx^n} = \phi(x)$ in equation (4), we get

$$(1 - x^2)\frac{d^2\phi}{dx^2} - 2x\frac{d\phi}{dx} + n(n + 1)\phi = 0$$

This is Legendre equation which has solution $\phi = \frac{d^n y}{dx^n}$. hence we may relate this solution $\phi(x)$ with Legendre polynomial $P_n(x)$ as

$P_n(x) = C\phi(x)$; where C is a constant. -------------- (6)

As $y = (x^2 - 1)^n = (x - 1)^n (x + 1)^n$; therefore differentiating both sides n times by Leibnitz theorem, we get

$$\frac{d^n y}{dx^n} = (x - 1)^n \frac{d^n}{dx^n}(x + 1)^n + {}^nC_1 n(x - 1)^{n-1} \frac{d^{n-1}}{dx^{n-1}}(x + 1)^n + ------$$

$$------ {}^nC_{n-1} n(x + 1)^{n-1} \frac{d^{n-1}}{dx^{n-1}}(x - 1)^n + {}^nC_n (x + 1)^n \frac{d^n}{dx^n}(x - 1)^n$$

Now putting x=1 on both sides of the above equation, all the terms in R.H.S. due to the factor (x-1) except the last term vanish and keeping in mind that $\frac{d^n}{dx^n}(x - 1)^n$=n!; we get

$$\left(\frac{d^n y}{dx^n}\right)_{x=1} = 2^n n! \text{ ------ (7)}$$

Substituting x=1 in equation (6); we get

$P_n(x) = C[\phi(x)]_{x=1} = C\ 2^n n!$; Using equation (7)

$$C = \frac{P_n(1)}{2^n n!} = \frac{1}{2^n n!} \quad (since\ P_n(1) = 1)$$

Substituting this value of C in equation (6), we get

$$P_n(x) = \frac{1}{2^n n!}\phi(x) = \frac{1}{2^n n!} \frac{d^n y}{dx^n} = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n \text{ ----- (8)}$$

This is Rodrigue's formula [since $y = (x^2 - 1)^n$]

Deductions: It is convenient to deduce the values of $P_0(x), P_1(x), P_2(x), P_3(x)$ $----$etc. by using Rodrigue's formula. Substituting n=0,1,2,3--- successively in equation (8), we obtain

$$P_0(x) = \frac{1}{2^0 0!} = 1 \qquad (since\ 0! = 1)$$

$$P_1(x) = \frac{1}{2^1 1!} \frac{d}{dx}(x^2 - 1) = \frac{1}{2!}(2x) = 2$$

$$P_2(x) = \frac{1}{2^2 2!} \frac{d^2}{dx^2}(x^2 - 1)^2 = \frac{1}{8}\frac{d}{dx}\{2(x^2 - 1)\ .2x\}$$

$$= \frac{1}{2}\frac{d}{dx}\{x(x^2 - 1)\} = \frac{1}{2}\frac{d}{dx}(x^3 - x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2^3 3!} \frac{d^3}{dx^3}(x^2 - 1)^3 = \frac{3x^3 - x}{2}$$

Proceedings as above we may deduce the values of $P_4(x), P_5(x)\ etc$