Formulas used

1. $log\,(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ------$
2. $log\,(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} -----$
3. $\int udv = uv - \int vdu$ (Integration by parts)
4. $\int (3x^3 - 2x)^{-1} = log\,(3x^3 - 2x) \times \frac{1}{\frac{d}{dx}\{(3x^3-2x)\}} = \frac{1}{9x^2-2}\,log\,(3x^3 - 2x)$

Orthogonal Properties of Legendre's Polynomials

To know that

$$\int_{-1}^{+1} P_m(x)P_n(x)dx = 0 \quad for\; m \neq n \quad \text{......................(a)}$$

$$\int_{-1}^{+1} [P_n(x)]^2 dx = \frac{2}{2n+1} \quad \text{......................(b)}$$

The above two equations may be written equivalently in the form of a single equation using **Kronecker Delta Symbol** δmn

(δmn =0 if m $\neq$ n and δmn =1 if m=n**)** as

$$\int_{-1}^{+1} P_m(x)P_n(x)dx = \frac{2}{2n+1}\,\delta mn \quad \text{.....................(c)}$$

Proof. (a) Legendre equation is

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

This may be expressed as

$$\frac{d}{dx}\{(1-x^2)\frac{dy}{dx}\} + n(n+1)y = 0 \text{......................(1)}$$

Also $P_n(x)$ and $P_m(x)$ are solutions of Legendre equation (1), therefore

$$\frac{d}{dx}\{(1-x^2)\frac{dP_n}{dx}\} + n(n+1)P_n = 0 \text{ ....(2)}$$

$$\frac{d}{dx}\{(1-x^2)\frac{dP_m}{dx}\} + m(m+1)P_m = 0 \text{ ....(3)}$$

And Multiplying equation (2) by P_m and (3) by P_n and subtracting, We get

$$P_m\frac{d}{dx}\{(1-x^2)\frac{dP_n}{dx}\} - P_n\frac{d}{dx}\{(1-x^2)\frac{dP_m}{dx}\} + P_mP_n[n(n+1) - m(m+1)] = 0$$

Integrating above equation between given limits, we get

$$n(n+1) - m(m+1) = n^2 + n - m^2 - m = n^2 - m^2 + n - m = (n-m)(n+m+1)$$

$$\int_{-1}^{+1} P_m \frac{d}{dx}\{(1-x^2)\frac{dP_n}{dx}\}dx - \int_{-1}^{+1} P_n \frac{d}{dx}\{(1-x^2)\frac{dP_m}{dx}\}dx + (n-m)(n+m+1)\int_{-1}^{+1} P_m P_n dx = 0$$

Integrating by parts, we get

$$[P_m(1-x^2)\frac{dP_n}{dx}]_{-1}^{+1} - \int_{-1}^{+1} \frac{dP_m}{dx}\{(1-x^2)\frac{dP_n}{dx}\}dx - [P_n(1-x^2)\frac{dP_m}{dx}]_{-1}^{+1} + \int_{-1}^{+1} \frac{dP_n}{dx}\{(1-x^2)\frac{dP_m}{dx}\}dx$$

$$+ (n-m)(n+m+1)\int_{-1}^{+1} P_m P_n dx = 0$$

Or

$$0 - \int_{-1}^{+1} \frac{dP_m}{dx}\{(1-x^2)\frac{dP_n}{dx}\}dx - 0 + \int_{-1}^{+1} \frac{dP_n}{dx}\{(1-x^2)\frac{dP_m}{dx}\}dx + (n-m)(n+m+1)\int_{-1}^{+1} P_m P_n dx = 0$$

or $$(n-m)(n+m+1)\int_{-1}^{+1} P_m P_n dx = 0.$$

If n≠m; we get $\int_{-1}^{+1} P_m(x)P_n(x)\, dx = 0$ …………………..(4).

From generating function of Legendre Polynomial, we have

$$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x)$$

Squaring both sides we get

$$(1-2xz+z^2)^{-1} = \sum_{m=0}^{\infty} z^m P_m(x) \sum_{n=0}^{\infty} z^n P_n(x)$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} z^m z^n p_m(x)\, P_n(x)$$

$$= \sum_{n=0}^{\infty} z^{2n} [P_n(x)]^2 + \sum_{m=0}^{\infty}\sum_{\substack{n=0 \\ m \neq n}}^{\infty} z^{m+n} p_m(x)\, P_n(x)$$

Integrating both sides with respect to x between the limits -1 to +1. We get

$$\int_{-1}^{+1} (1 - 2xz + z^2)^{-1} dx = \sum_{n=0}^{\infty}\int_{-1}^{+1} z^{2n}[P_n(x)]^2 dx + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\int_{-1}^{+1} z^{m+n} P_m(x)P_n(x)\, dx$$

But from equation (4) $\int_{-1}^{+1} P_m(x) P_n(x)\, dx = 0$ for m≠n; therefore

$$\int_{-1}^{+1} (1 - 2xz + z^2)^{-1} dx = \sum_{n=0}^{\infty} z^{2n} \int_{-1}^{+1} [P_n(x)]^2 dx$$

$$or \qquad -\frac{1}{2z} [log(1 - 2xz + z^2]_{-1}^{+1} = \sum_{n=0}^{\infty} z^{2n} \int_{-1}^{+1} [P_n(x)]^2 dx$$

$$or \qquad -\frac{1}{2z} [log(1 - z)^2 - log(1 + z)^2] = \sum_{n=0}^{\infty} z^{2n} \int_{-1}^{+1} [P_n(x)]^2 dx$$

$$or \qquad -\frac{1}{2z} [log(1 - z)^2 - log(1 + z)^2] = \sum_{n=0}^{\infty} z^{2n} \int_{-1}^{+1} [P_n(x)]^2 dx$$

$$or \qquad -\frac{1}{z} [log(1 - z) - log(1 + z)] = \sum_{n=0}^{\infty} z^{2n} \int_{-1}^{+1} [P_n(x)]^2 dx$$

$$or \qquad \frac{1}{z} [log(1 + z) - log(1 - z)] = \sum_{n=0}^{\infty} z^{2n} \int_{-1}^{+1} [P_n(x)]^2 dx$$

$$or \qquad \frac{1}{z} [(z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots\ldots) - (- z - \frac{z^2}{2} - \frac{z^3}{3} - \ldots\ldots)] = \sum_{n=0}^{\infty} z^{2n} \int_{-1}^{+1} [P_n(x)]^2 dx$$

$$or \qquad \frac{2}{z} [z + \frac{z^3}{3} + \frac{z^5}{5} + \ldots\ldots + \frac{z^{2n+1}}{2n+1} + \ldots] = \sum_{n=0}^{\infty} z^{2n} \int_{-1}^{+1} [P_n(x)]^2 dx$$

$$or \qquad 2 [1 + \frac{z^2}{3} + \frac{z^3}{5} + \ldots\ldots + \frac{z^{2n}}{2n+1} + \ldots] = \sum_{n=0}^{\infty} z^{2n} \int_{-1}^{+1} [P_n(x)]^2 dx$$

Equating coefficients of z^{2n} on either side, we get

$$\int_{-1}^{+1} [P_n(x)]^2 dx = \frac{2}{2n+1} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Combining equations (4) and (5),we may write

$$\int_{-1}^{+1} p_m(x)\, P_n(x) dx = \frac{2}{2n+1} \delta mn \qquad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$