Recurrence formulae of $P_n(x)$:

$1a)\quad nPn = (2n-1)xP_{n-1} - (n-1)\ P_{n-2}$

$1b)\quad (n+1)P_{n+1} = (2n+1)xP_n - nP_{n-1}$

$2)\quad nP_n = xP'_n - P'_{n-1}$

$3a)\quad P'_{n+1} - P'_{n-1} = (2n+1)P_n$

$3b)\quad P'_n - P'_{n-2} = (2n-1)P_{n-1}$

$4a)\quad P'_{n+1} = xP'_n + (n+1)\ P_n$

$4b)\quad P'_n = xP'_{n-1} + n\ P_{n-1}$

$5)\quad (1-x^2)\ P'_n = n(P_{n-1} - xP_n)$

$6)\quad (1-x^2)\ P'_n = (n+1)(xP_n - P_{n+1})$

At a Glance

eqn	From	
1a	Generating function of legendre polynomial	Coefficients of z^{n-1}
1b		$n \rightarrow n+1$
2	Generating function of legendre polynomial	1. z diff.---(1) 2. x diff ---(2) and (2)/(1) 3. Coefficients of z^n
3	1b, 2	
3b		$n \rightarrow n-1$
4	1b, 2	
4b		$n \rightarrow n-1$
5	4b, 2	
6	1b, 5	

Formula (1): $nPn = (2n - 1)xP_{n-1} - (n - 1)\ P_{n-2}$ ……………….(1)

Proof : From the generating function of $P_n(x)$

$$(1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x)$$

Differentiating above equation With respect to Z, we get

$$-\frac{1}{2}(1 - 2xz + z^2)^{-3/2}(- 2x + 2z) = \sum_{n=0}^{\infty} n\ z^{n-1} P_n(x)$$

$$(x - z)(1 - 2xz + z^2)^{-3/2} = \sum_{n=0}^{\infty} n\ z^{n-1} P_n(x)$$

Multiplying both sides with $(1 - 2xz + z^2)$; we get

$$(x - z)(1 - 2xz + z^2)^{-1/2} = (1 - 2xz + z^2) \sum_{n=0}^{\infty} n\ z^{n-1} P_n(x)$$

$$(x - z) \sum_{n=0}^{\infty} z^n P_n(x) = (1 - 2xz + z^2) \sum_{n=0}^{\infty} n\ z^{n-1} P_n(x) \text{……………………..} \quad \text{(A)}$$

$$x\sum_{n=0}^{\infty} z^n P_n(x) - z\sum_{n=0}^{\infty} z^n P_n(x) = \sum_{n=0}^{\infty} n\ z^{n-1} P_n(x) - 2xz\sum_{n=0}^{\infty} n\ z^{n-1} P_n(x) + z^2 \sum_{n=0}^{\infty} n\ z^{n-1} P_n(x)$$

$$x\sum_{n=0}^{\infty} z^n P_n(x) - \sum_{n=0}^{\infty} z^{n+1} P_n(x) = \sum_{n=0}^{\infty} n\ z^{n-1} P_n(x) - 2x\sum_{n=0}^{\infty} n\ z^n P_n(x) + \sum_{n=0}^{\infty} n\ z^{n+1} P_n(x)$$

Equating coefficients of z^{n-1} from both sides, we get

$$xP_{n-1} - P_{n-2} = nP_n - 2x(n - 1)P_{n-1} + (n - 2)P_{n-2}$$

$$nP_n = xP_{n-1} - P_{n-2} + 2x(n - 1)P_{n-1} - (n - 2)P_{n-2}$$

$$nP_n = x(2n - 2 + 1)P_{n-1} - (n - 2 + 1)P_{n-2}$$

$$nP_n = (2n - 1)xP_{n-1} - (n - 1)P_{n-2}$$

This is the first recurrence relation. This ,may be written in alternative form by submitting (n+1) for n in this relation or equating the coefficients of z^n form both sides of equation (A) as

$$(n + 1)P_{n+1} = (2n + 1)xP_n - nP_{n-1} \text{……………………..} \quad \text{(1b)}$$

Formula (2) : $nP_n = x\frac{dP_n}{dx} - \frac{dP_{n-1}}{dx}$

$$nP_n = xP'_n - P'_{n-1}$$

Where dashes denote differentiation with respect to x,

Proof : We have

$$(1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x) \text{………………} \quad \text{(B)}$$

Differentiating this equation with respect to z,we get

$$-\frac{1}{2}(1-2xz+z^2)^{-3/2}(-2x+2z)=\sum_{n=0}^{\infty} n\, z^{n-1} P_n(x)$$

$$(x-z)(1-2xz+z^2)^{-3/2}=\sum_{n=0}^{\infty} n\, z^{n-1} P_n(x) \qquad \text{.....................(C)}$$

Now differentiating eq(B) wrt x, we get

$$-\frac{1}{2}(1-2xz+z^2)^{-3/2}(-2z)=\sum_{n=0}^{\infty} z^n \frac{dp_n(x)}{dx}$$

$$z(1-2xz+z^2)^{-3/2}=\sum_{n=0}^{\infty} z^n \frac{dp_n(x)}{dx} \qquad \text{..................... (D)}$$

Dividing equation (C) by (D); we get

$$\frac{(x-z)}{z}=\frac{\sum_{n=0}^{\infty} n\, z^{n-1} Pn(x)}{\sum_{n=0}^{\infty} z^n\left(\frac{dp_n(x)}{dx}\right)}$$

$$i.e. \qquad (x-z)\sum_{n=0}^{\infty} z^n \frac{dp_n(x)}{dx}=z\sum_{n=0}^{\infty} n\, z^{n-1} P_n(x)$$

$$x\sum_{n=0}^{\infty} z^n \frac{dp_n(x)}{dx}-z\sum_{n=0}^{\infty} z^n \frac{dp_n(x)}{dx}=z\sum_{n=0}^{\infty} n\, z^{n-1} P_n(x)$$

$$x\sum_{n=0}^{\infty} z^n \frac{dp_n(x)}{dx}-\sum_{n=0}^{\infty} z^{n+1} \frac{dp_n(x)}{dx}=\sum_{n=0}^{\infty} n\, z^{n} P_n(x) \qquad \text{............ (E)}$$

Equating coefficients of z^n form both sides of (E), we get

$$x\frac{dp_n(x)}{dx}-\frac{dp_{n-1}(x)}{dx}=nP_n(x)$$

$$or \quad nP_n(x)=x\frac{dp_n(x)}{dx}-\frac{dp_{n-1}(x)}{dx}$$

or $\quad nP_n = xP'_n - P'_{n-1} \qquad$ ....(2)

Formula (3) : $P_{n-1}(x) - P'_{n-1} = (2n+1)P_n(x)$

Proof. From relation (1b), we have

$$(n+1)P_{n+1}=(2n+1)xP_n-nP_{n-1}$$

Differentiating with respect to x, we get

$$(n+1)P'_{n+1}=(2n+1)xP'_n+(2n+1)P_n-nP'_{n-1} \qquad \text{............(F)}$$

From relation (2) ; $\quad nP_n = xP'_n - P'_{n-1}$

$xP'_n = nP_n + P'_{n-1}$ ....(G)

Substituting this value of xP'_n in equation (F); we get

$(n + 1)P'_{n+1} = (2n + 1)xP'_n + (2n + 1)P_n - nP'_{n-1}$

$(n + 1)P'_{n+1} = (2n + 1)(nP_n + P'_{n-1}) + (2n + 1)P_n - nP'_{n-1}$

$(n + 1)P'_{n+1} = (2n + 1)nP_n + (2n + 1)P'_{n-1} + (2n + 1)P_n - nP'_{n-1}$

$(n + 1)P'_{n+1} = (2n + 1)(n + 1)P_n + (2n + 1 - n)P'_{n-1}$

$(n + 1)P'_{n+1} = (2n + 1)(n + 1)P_n + (n + 1)P'_{n-1}$

$P'_{n+1} = (2n + 1)P_n + P'_{n-1}$

$P'_{n+1} - P'_{n-1} = (2n + 1)P_n$ ………………..(3a)

This is Ⅲ recurrence relation. This may be expressed in alternative form if we substitute (n-1) for n in above relation, i,e

$$P'_n - P'_{n-2} = (2n - 1)P_{n-1} \quad \text{…………………. (3b)}$$

Formula (4) : $P'_{n+1} - xP'_n = (n + 1)P_n$

From relation (1b)

$$(n + 1)P_{n+1} = (2n + 1)xP_n - nP_{n-1}$$

Differentiating with respect to x. We get

$(n + 1)P'_{n+1} = (2n + 1)P_n + (2n + 1)xP'_n - nP'_{n-1}$

$= (2n + 1)P_n + (n + n + 1)xP'_n - nP'_{n-1}$

$= (2n + 1)P_n + (n + 1)xP'_n + nxP'_n - nP'_{n-1}$

$= (2n + 1)P_n + (n + 1)xP'_n + n(xP'_n - P'_{n-1})$

$= (2n + 1)P_n + (n + 1)xP'_n + n(nP_n)$ [using relation (2)]

$= (n + 1)xP'_n + \{(2n + 1) + n^2\}P_n$

$i.e \quad (n + 1)P'_{n+1} = (n + 1)xP'_n + (n + 1)^2P_n$

$$P'_{n+1} = xP'_n + (n + 1)\ P_n \quad \text{……………………. (4a)}$$

This is Ⅳ recurrence relation. This may be expressed in alternative for if we substitute (n-1) fo n in (4a), i,e

$$P'_n = xP'_{n-1} + n\ P_{n-1} \quad \text{……………………. (4b)}$$

Formula (5) : $(1 - x^2)\ P'_n = n(P_{n-1} - xP_n)$

From relation (4b) ; $P'_n = xP'_{n-1} + n\ P_{n-1}$

$n\ P_{n-1} = P'_n - xP'_{n-1}$………………..…. (H)

From relation (2) ;

$$nP_n = xP'_n - P'_{n-1} \quad \text{…………………. (I)}$$

Multiplying equation (I) by x and then subtracting from (H) ;

$$xnP_n - n\ P_{n-1} = x^2P'_n - xP'_{n-1} - P'_n + xP'_{n-1}$$

$$n(xP_n - P_{n-1}) = (x^2 - 1)P'_n$$

$$(1 - x^2)\ \ P'_n = n(P_{n-1} - xP_n)\ \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Formula (6) : $(1 - x^2)\ \ P'_n = (n + 1)(xP_n - P_{n+1})$

From relation (1b)

$$(n + 1)P_{n+1} = (2n + 1)xP_n - nP_{n-1}$$

$$= (n + n + 1)xP_n - nP_{n-1}$$

$$= (n + 1)xP_n + nxP_n - nP_{n-1}$$

$$= (n + 1)xP_n - n(P_{n-1} - xP_n)$$

$$= (n + 1)xP_n - (1 - x^2)\ \ P'_n$$ **(**using relation 5**)**

$$(1 - x^2)\ \ P'_n = (n + 1)(xP_n - P_{n+1})\text{-----------}(6)$$