Laguerre's Differential Equation and Laguerre Polynomials

Laguerre differential equation is

$$x \frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + ny = 0 \qquad \ldots\ldots\ldots\ldots (1)$$

Let the series solutions of above equation be

$$y = \sum_{r=0}^{\infty} a_r x^{k+r} \qquad \ldots\ldots\ldots\ldots\ldots (2)$$

So that $\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (k+r) x^{k+r-1}$

and $\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-2}$

Substituting these values in (1) ; we get

$$x\sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-2} + (1-x)\sum_{r=0}^{\infty} a_r (k+r) x^{k+r-1} + n\sum_{r=0}^{\infty} a_r x^{k+r} = 0$$

$$x\sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-2} + \sum_{r=0}^{\infty} a_r (k+r) x^{k+r-1} - x\sum_{r=0}^{\infty} a_r (k+r) x^{k+r-1} + n\sum_{r=0}^{\infty} a_r x^{k+r} = 0$$

$$\sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-1} + \sum_{r=0}^{\infty} a_r (k+r) x^{k+r-1} - \sum_{r=0}^{\infty} a_r (k+r) x^{k+r} + n\sum_{r=0}^{\infty} a_r x^{k+r} = 0$$

or $$\sum_{r=0}^{\infty} a_r [(k+r)^2 x^{k+r-1} - (k+r-n) x^{k+r}] = 0 \ldots\ldots\ldots\ldots (3)$$

Equation (3) is an identity ; therefore the various powers of x must be zero. Equating to zero the coefficients of lowest power of x i.e,.. (putting r=0) we get

$$a_0 k^2 = 0$$

as $a_0 \neq 0$, being the coefficient of first term of the series ; therefore

$$k = 0 \qquad \ldots\ldots\ldots\ldots (4)$$

Now equating to zero, the coefficient of general term x^{k+r} ; we get

$a_{r+1} (k+r+1)^2 - a_r (k+r-n) = 0$ or $a_{r+1} = \frac{k+r-n}{(k+r+1)^2} a_r$

As k = 0; we have $a_{r+1} = \frac{r-n}{(r+1)^2} a_r \qquad \ldots\ldots\ldots\ldots (5)$

Substituting r =0, 1, 2, 3……. Etc in given equation (5); we get

For r=0 $\qquad a_1 = -\frac{n}{1} a_0 = (-1)\, n a_0$

For r=1 $a_2 = \frac{(1-n)}{2^2} a_1 = -\frac{(n-1)}{2^2} X\ (-1)\ n a_0 = (-1)^2 \frac{n(n-1)}{(2!)^2} a_0$

For r=2 $a_3 = \frac{(2-n)}{2^2} a_2 = (-1)^3 \frac{n(n-1)(n-2)}{(3!)^2} a_0$

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For r $a_r = (-1)^r \frac{n(n-1)(n-2) \ldots\ldots (n-r+1)}{(r!)^2} a_0$

Therefore from equation (2) ; we have (for k = 0)

$$y = \sum_{r=0}^{\infty} a_r x^{k+r} = \sum_r a_r x^r$$

$$= a_0 + a_1 x^1 + a_2 x^2 + \ldots\ldots + a_r x^r + \ldots\ldots$$

$$= a_0 \left[1 - nx + \frac{n(n-1)}{(2!)^2} x^2 + \ldots\ldots + (-1)^r \frac{n(n-1)(n-2) \ldots\ldots (n-r+1)}{(r!)^2} x^r + \ldots\ldots \right]$$

$$= a_0 \sum_{r=0}^{\infty} (-1)^r \frac{n(n-1)(n-2) \ldots\ldots (n-r+1)}{(r!)^2} x^r$$

$$= a_0 \sum_{r=0}^{\infty} \frac{(-1)^r n!}{(r!)^2 (n-r)!} x^r \qquad \ldots\ldots\ldots\ldots (6)$$

In case *n* is a positive integer and $a_0 = 1$ and the series terminate after n^{th} degree term, the Solution (6) is said to be **Laguerre polynomial of degree n** and is denoted by

$$L_n(x) = \sum_{r=0}^{n} \frac{(-1)^r n!}{(r!)^2 (n-r)!} x^r \qquad \ldots\ldots\ldots\ldots (7)$$

Then the solution of Laguerre equation for *n* to be a positive integer is

$$y = A\, L_n(x) \qquad \ldots\ldots\ldots\ldots (8)$$