Generating Function for Laguerre Polynomials

Theorem. The generating function for Laguerre Polynomials is

$$f(x, z) = \frac{e^{-xz/(1-z)}}{1-z} = \sum_{n=0}^{\infty} L_n(x)\, z^n \quad \text{for } |z| < 1$$

Proof : we have

$$\frac{1}{1-z} e^{-xz/(1-z)} = \frac{1}{1-z} \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{-xz}{1-z}\right)^r$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \frac{z^r}{(1-z)^{r+1}} x^r$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} z^r x^r (1 - z)^{-(r+1)}$$

$$(1 - x)^{-n} = 1 + nx + \frac{n(n+1)}{2!} x^2 + \frac{n(n+1)(n+2)}{3!} x^3 + \cdots + \frac{n(n+1)(n+2)\cdots(n+r-1)}{r!} x^r + \cdots$$

$$= \sum_{r=0}^{\infty} \frac{n(n+1)(n+2)\cdots(n+r-1)}{r!} x^r$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} z^r x^r \sum_{s=0}^{\infty} \frac{(r+1)(r+2) \ldots\ldots (r+s)}{s!} z^s$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} z^r x^r \sum_{s=0}^{\infty} \frac{(r+s)!}{r!s!} z^s$$

$$= \sum_{r,s=0}^{\infty} \frac{(-1)^r (r+s)!}{(r!)^2 s!} x^r z^{r+s} \quad \ldots\ldots\ldots\ldots (1)$$

The coefficient of z^n (for fixed value of r) on R.H.S is obtained by putting r+s = n i,e. s = n-r and is given by

$$(-1)^r \frac{n!}{r!\,(n-r)!} x^r$$

The net coefficient of z^n is obtained by summing over all allowed values of r.

As s = n-r and $s \geq 0$ ∴ $n-r \geq 0$ or $r \leq n$

Hence net coefficient of z^n on R.H.S of (1) is

$$\sum_{r=0}^{\infty} \frac{(-1)^r n!}{(r!)^2 (n-r)!} x^r = L_n(x) \qquad \ldots\ldots\ldots\ldots. (2)$$

Hence we may write

$$\frac{e^{-xz/(1-z)}}{1-z} = \sum_{n=0}^{\infty} L_n(x)\, z^n$$

Thus the function $\frac{e^{-xz/(1-z)}}{1-z}$ generates all the Laguerre's polynomials and hence it is called the **generating function of Laguerre polynomials.**

Rodrigue's formula of Laguerre polynomials

The Rodrigue's formula of Laguerre polynomials is

$$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n}(x^n e^{-x}), \text{ } n \text{ is an integer.}$$

Proof : we have

$$\frac{1}{1-z} e^{-xz/(1-z)} = \sum_{n=0}^{\infty} L_n(x)\, z^n$$

$$[1 - \frac{1}{(1-z)}]x = [\frac{1-z-1}{(1-z)}]x = \frac{-xz}{(1-z)}$$

or

$$(1-z)^{-1} e^{[1-1/(1-z)]x} = \sum_{n=0}^{\infty} L_n(x)\, z^n$$

Differentiating this equation n times with respect to z ; we get

$$\frac{d^n}{dz^n}(1-z)^{-1}\, e^{[1-1/(1-z)]x} = \frac{d^n}{dz^n} \sum_{n=0}^{\infty} L_n(x)\, z^n$$

$$e^x \frac{d^n}{dz^n}\{(1-z)^{-1}\, e^{-x/(1-z)}\} = n!\, L_n(x) + \frac{(n+1)!}{1!} L_{n+1}(x).z + \frac{(n+2)!}{2!} L_{n+2}(x).z^2 + \ldots\ldots\ldots\ldots. (1)$$

Now $$\lim_{z \to 0} \frac{d}{dz}[(1-z)^{-1}\, e^{-x/(1-z)}] = \lim_{z \to 0} \frac{1-x-z}{(1-z)^3} e^{-x/(1-z)} = (1-x)\, e^{-x} = \frac{d}{dx}(xe^{-x})$$

$$\frac{d}{dz}[(1-z)^{-1}e^{-\frac{x}{1-z}}] = (-1)(1-z)^{-2}(-1)e^{-\frac{x}{1-z}} + (1-z)^{-1}e^{-\frac{x}{1-z}}(-x)(-1)(1-z)^{-2}(-1)$$
$$= \frac{e^{-\frac{x}{1-z}}}{(1-z)^2} - \frac{xe^{-\frac{x}{1-z}}}{(1-z)^3} = \frac{e^{-\frac{x}{1-z}}}{}\frac{(1-z-x)}{(1-z)^3}$$

Similarly, $\lim_{z\to 0}\{\frac{d^2}{dz^2}(1-z)^{-1}\, e^{-x/(1-z)}\} = \frac{d^2}{dx^2}(x^2e^{-x})$

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By induction $\lim_{z\to 0}\{\frac{d^n}{dz^n}(1-z)^{-1}\, e^{-x/(1-z)}\} = \frac{d^n}{dx^n}(x^ne^{-x})$

Hence equation (1) in the limit $z\to 0$ gives

$$e^x\frac{d^n}{dx^n}(x^ne^{-x}) = n!\,L_n(x)$$

$$\therefore \qquad L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(x^ne^{-x}) \qquad \text{…………… (2)}$$

Which is the required **Rodrigue's representation of Laguerre's polynomials.**

This representation is specifically useful in finding the Laguerre polynomials. Substituting n=0, 1 ,2 ,3 …….etc in (2) ; we get

$L_0(x) = \frac{e^x}{0!}(x^0e^{-x}) = 1$

$L_1(x) = \frac{e^x}{1!}\frac{d}{dx}(xe^{-x}) = e^x(e^{-x}- xe^{-x}) = e^x.e^{-x}(1- x)= 1-x$

$L_2(x) = \frac{e^x}{2!}\frac{d^2}{dx^2}(x^2e^{-x}) = = \frac{1}{2!}(2 - 4x + x^2)$

$$L_2(x) = \frac{e^x}{2!}\frac{d^2}{dx^2}(x^2e^{-x}) = \frac{e^x}{2!}\frac{d}{dx}[e^{-x}.2x + x^2(-1)e^{-x}] = \frac{e^x}{2!}\frac{d}{dx}(e^{-x}.2x - x^2e^{-x})$$
$$= \frac{e^x}{2!}\frac{d}{dx}[e^{-x}(2x - x^2)] = \frac{e^x}{2!}\{e^{-x}(2 - 2x) - [e^{-x}(2x - x^2)]\}$$
$$= \frac{e^x}{2!}e^{-x}(2 - 2x - 2x + x^2) = \frac{1}{2!}(2 - 4x + x^2)$$

Similarly $L_3(x) = \frac{1}{3!}(6-18x+9x^2-x^3)$

And $L_4(x) = \frac{1}{4!}(24-96x+72x^2-16x^3+x^4)$ etc.