Recurrence Relations for Laguerre Polynomials

(I) $(n+1)\, L_{n+1}(x) = (2n+1-x)\, L_n(x) - n\, L_{n-1}(x)$

Proof : we have from the property of generating function of Laguerre Polynomials

$$\frac{e^{-xz/(1-z)}}{1-z} = \sum_{n=0}^{\infty} L_n(x) z^n \qquad \text{............ (i)}$$

Differentiating both sides with respect to z; we get

$$\frac{1}{(1-z)^2} e^{-xz/(1-z)} + \frac{1}{(1-z)} e^{-xz/(1-z)} \left\{\frac{-x}{(1-z)^2}\right\} = \sum_{n=0}^{\infty} L_n(x)\, n.z^{n-1}$$

$$\frac{1}{(1-z)} \sum_{n=0}^{\infty} L_n(x) z^n - \frac{x}{(1-z)^2} \sum_{n=0}^{\infty} L_n(x) z^n = \sum_{n=0}^{\infty} L_n(x) n z^{n-1}$$

Multiplying throughout by $(1-z)^2$ and using (i); we get

$$(1-z) \sum_{n=0}^{\infty} L_n(x)\, z^n - x \sum_{n=0}^{\infty} L_n(x)\, z^n = (1-z)^2 \sum_{n=0}^{\infty} L_n(x)\, n.z^{n-1}$$

$$(1-x-z) \sum_{n=0}^{\infty} L_n(x)\, z^n = (1-2z+z^2) \sum_{n=0}^{\infty} L_n(x)\, n.z^{n-1}$$

$$(1-x) \sum_{n=0}^{\infty} L_n(x)\, z^n - \sum_{n=0}^{\infty} L_n(x)\, z^{n+1} = \sum_{n=0}^{\infty} L_n(x)\, n.z^{n-1} - 2n \sum_{n=0}^{\infty} L_n(x)\, z^n + \sum_{n=0}^{\infty} L_n(x)\, n.z^{n+1}$$

Comparing coefficients of z^n on either side , we get

$$(1-x)\, L_n(x) - L_{n-1}(x) = (n+1)L_{n+1} - 2n\, L_n(x) + (n-1)\, L_{n-1}(x)$$

$$(2n+1-x)L_n(x) - [1+(n-1)]\, L_{n-1}(x) = (n+1)\, L_{n+1}(x)$$

$$\therefore \quad (n+1)\, L_{n+1}(x) = (2n+1-x)\, L_n(x) - n\, L_{n-1}(x) \qquad \text{............(i)}$$

(II) $x\, L'_n(x) = n\, L_n(x) - n\, L_{n-1}(x)$

Proof : we have

$$\frac{e^{-xz/(1-z)}}{1-z} = \sum_{n=0}^{\infty} L_n(x)\, z^n$$

Differentiating both sides with respect to x, we get

$$\frac{1}{(1-z)} e^{-xz/(1-z)} \cdot \left(\frac{-z}{1-z}\right) = \sum_{n=0}^{\infty} L'_n(x)\, z^n$$

or
$$-z \frac{1}{(1-z)} e^{-xz/(1-z)} = (1-z) \sum_{n=0}^{\infty} L'_n(x)\, z^n$$

or
$$-z \sum_{n=0}^{\infty} L_n(x)\, z^n = (1-z) \sum_{n=0}^{\infty} L'_n(x)\, z^n$$

or
$$- \sum_{n=0}^{\infty} L_n(x)\, z^{n+1} = \sum_{n=0}^{\infty} L'_n(x)\, z^n - \sum_{n=0}^{\infty} L'_n(x)\, z^{n+1}$$

Equating coefficients of z^n on either side, we get

$$- L_{n-1}(x) = L'_n(x) - L'_{n-1}(x)$$

or
$$L'_n(x) = L'_{n-1}(x) - L_{n-1}(x) \quad \text{................. (ii)}$$

Relation (i) is

$$(n+1)\, L_{n+1}(x) = (2n+1-x)\, L_n(x) - n\, L_{n-1}(x)$$

Differentiating recurrence relation (i) with respect to x, we get

$$(n+1)\, L'_{n+1}(x) = (2n+1-x)\, L'_n(x) - L_n(x) - n\, L'_{n-1}(x) \quad \text{........... (iii)}$$

Replacing *n* by (n+1) in (ii); we get

$$L'_{n+1}(x) = L'_n(x) - L_n(x) \quad \text{.................. (iv)}$$

From (ii)

$$L'_{n-1}(x) = L'_n(x) + L_{n-1}(x) \quad \text{........... (v)}$$

or values of $L'_{n+1}(x)$ and $L'_{n-1}(x)$ from (iv) and (v) in (iii) we get

$$(n+1)\, L'_{n+1}(x) = (2n+1-x)\, L'_n(x) - L_n(x) - n\, L'_{n-1}(x)$$

$$(n+1)[L'_n(x) - L_n(x)] = (2n+1-x)\, L'_n(x) - L_n(x) - n\, [L'_n(x) + L_{n-1}(x)]$$

Rearranging the coefficients

$$(n+1)L'_n(x) - (n+1)L_n(x) = (2n+1-x)\, L'_n(x) - L_n(x) - n\, L'_n(x) - nL_{n-1}(x)$$

$(n+1-2n-1+x+n)L'_n(x) = (n+1-1)\, L_n(x) - nL_{n-1}(x)$

$x\, L'_n(x) = n\, L_n(x) - n\, L_{n-1}(x)$ --------- (2)

(iii) $L'_n(x) = -\sum_{r=0}^{n-1} L_r(x)$

Proof : we have

$$\frac{1}{(1-z)} e^{-xz/(1-z)} = \sum_{n=0}^{\infty} L_n(x)\, z^n$$

Differentiating wrt to x, we get

$$\left(-\frac{z}{1-z}\right) \frac{1}{(1-z)} e^{-xz/(1-z)} = \sum_{n=0}^{\infty} L'_n(x)\, z^n$$

$$\sum_{n=0}^{\infty} L'_n(x)\, z^n = -z\,(1-z)^{-1} \sum_{r=0}^{\infty} L_r(x)\, z^r$$

$$= z[1+z+z^2+ \ldots + z^s+ \ldots] \left[\sum_{r=0}^{\infty} L_r(x)\, z^r\right]$$

$$= -z \sum_{s=0}^{\infty} z^s \sum_{r=0}^{\infty} L_r(x)\, z^r$$

$$= -\sum_{r,s=0}^{\infty} L_r(x)\, z^{r+s+1.} \qquad \ldots\ldots\ldots\ldots\ldots \text{(vii)}$$

For the coefficient of z^n for fixed value of r is obtained by putting $r+s+1=n$ and is given by $-L_r(x)$
The net coefficient of z^n is obtained by summing over all allowed values of r.
As $r+s+1 = n$ or $s = n-r-1$ and $s \geq 0$

$\therefore\ n-r-1 \geq 0$ or $r \leq n-1$

coefficients of z^n on right hand side is $-\sum_{r=0}^{n-1} L_r(x)$

If we equate coefficients of z^n on either side of (vii); we get

$$L'_n(x) = -\sum_{r=0}^{n-1} L_r(x) \qquad \ldots\ldots\ldots\ldots \text{(3)}$$

Orthogonal Property to laguerre Polynomials

The *Laguerre polynomials* do not themselves form an orthogonal set.

However, the related set of functions

$$\phi_n(x) = e^{-(x/2)} L_n(x)$$

From an orthogonal set for the internal $0 \le x \le \infty$, i,e,.

$$\int_0^\infty \phi_m(x).\phi_n(x)\, dx = \int_0^\infty e^{-(x/2)} L_m(x).e^{-(x/2)} L_n(x)\, dx = \delta_{mn}$$

$$\int_0^\infty e^{-x} L_m(x).L_n(x)\, dx = \delta_{mn}$$

Proof : we have

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(x^n e^{-x}) \qquad \text{........(1)}$$

$$e^{-x} L_n(x) = \frac{1}{n!}\frac{d^n}{dx^n}(x^n e^{-x})$$

$$\int_0^\infty e^{-x} x^m L_n(x)\, dx = \int_0^\infty x^m \frac{1}{n!}\frac{d^n}{dx^n}(x^n e^{-x})\, dx$$

$$\int u\,dv = uv - \int v\,du$$

$$\int_0^\infty x^m \frac{1}{n!}\frac{d^n}{dx^n}(x^n e^{-x})dx = \frac{1}{n!}\int_0^\infty x^m \frac{d^n}{dx^n}(x^n e^{-x})dx$$

$$= \frac{1}{n!}\left[x^m \frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x})\right]_0^\infty - \frac{1}{n!}\int_0^\infty m x^{m-1} \frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x})dx$$

$$- \frac{m}{n!}\int_0^\infty x^{m-1} \frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x})dx$$

$$= - \frac{m}{n!}\left[x^{m-1} \frac{d^{n-2}}{dx^{n-2}}(x^n e^{-x})\right]_0^\infty + \frac{m}{n!}\int_0^\infty (m-1) x^{m-2} \frac{d^{n-2}}{dx^{n-2}}(x^n e^{-x})dx$$

$$= (-1)^m \frac{m!}{n!}\int_0^\infty \frac{d^{n-m}}{dx^{n-m}}(x^n e^{-x})\, dx \qquad \text{[on integration by parts]}$$

$$= (-1)^m \frac{m!}{n!}\int_0^\infty \frac{d^{n-m}}{dx^{n-m}}(x^n e^{-x})dx = (-1)^m \frac{m!}{n!}\left[\frac{d^{n-m-1}}{dx^{n-m-1}}(x^n e^{-x})\right]_0^\infty$$

$= 0$ if $n > m$ (since $e^{-\infty}=0$) .... (2a)

Similarly, we get, $\int_0^\infty e^{-x}x^n L_m(x)\, dx = 0$ if m>n …………(2b)

As L_n and L_m are polynomials of degree n and m in x respectively;
We get from (2a) and (2b).

$$\int_0^\infty e^{-x} L_m(x)L_n(x)\, dx = 0 \quad \text{if } m \neq n$$

Multiplying equation (2a) by $L_m(x)$ and (2b) by $L_n(x)$ and subtracting; we get

$$\int_0^\infty e^{-x}x^m L_n(x) L_m(x)\, dx - \int_0^\infty e^{-x}x^n L_m(x)L_n(x)\, dx=0$$

$$(x^m-x^n) \int_0^\infty e^{-x} L_n(x) L_m(x)\, dx =0$$

In case m=n, then it is obvious from equation (1) that the term of degree n in $L_n(x)$ is $\frac{(-1)^n x^n}{n!}$;

therefore if m=n; we have

$$\frac{d}{dx}(x^n e^{-x}) = [x^n(-1)e^{-x} + e^{-x} n\, x^{n-1})$$

$$\frac{d^2}{dx^2}(x^n e^{-x}) = \frac{d}{dx}(x^n(-1)e^{-x} + e^{-x} n\, x^{n-1})$$

$$= x^n(-1)^2 e^{-x} + (-1)e^{-x} n\, x^{n-1} + e^{-x} n\,(n-1)x^{n-2}$$

$$\frac{d^3}{dx^3}(x^n e^{-x})=$$

$$x^n(-1)^3 e^{-x} + (-1)^2 e^{-x} n\, x^{n-1} + (-1)^2 e^{-x} n\, x^{n-1} + (-1)e^{-x} n\,(n-1)x^{n-2} + - - -$$

$$\int_0^\infty e^{-x}[L_n(x)]^2 dx = \int_0^\infty e^{-x}\frac{e^x}{n!}x^n(-1)^n e^{-x}[L_n(x)]dx = \frac{(-1)^n}{n!}\int_0^\infty e^{-x} x^n \frac{e^x}{n!}\frac{d^n}{dx^n}(x^n e^{-x})dx$$

$$= \frac{(-1)^n}{(n!)^2}\int_0^\infty x^n \frac{d^n}{dx^n}(x^n e^{-x})dx$$

$$\int_0^\infty x^n \frac{d^n}{dx^n}(x^n e^{-x})dx = [x^n \frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x})]_0^\infty - \int_0^\infty \frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x}) n\, x^{n-1} dx$$

$$=-\int_0^\infty \frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x}) n\, x^{n-1} dx$$

$$=-\left[nx^{n-1}\frac{d^{n-2}}{dx^{n-2}}(x^n e^{-x})\right]_0^{\infty}+\int_0^{\infty}\frac{d^{n-2}}{dx^{n-2}}(x^n e^{-x})n(n-1)\,x^{n-2}dx$$

$$=(-1)^n n!\int_0^{\infty}(x^n e^{-x})dx$$

$$=(-1)^{2n}\frac{n!}{(n!)^2}\int_0^{\infty}x^n\,e^{-x}dx$$ (on integration by parts)

$$\int_0^{\infty}x^n\,e^{-x}dx=[(-1)x^n\,e^{-x}\,]_0^{\infty}-\int_0^{\infty}(-1)\,e^{-x}nx^{n-1}dx$$

$$=[(-1)^2e^{-x}nx^{n-1}]_0^{\infty}-\int_0^{\infty}(-1)\,e^{-x}n(n-1)x^{n-2}dx$$

=n!

$$=(-1)^{2n}\frac{n!}{(n!)^2}n!\quad=1$$

Combining (3) and (4); we get

$$\int_0^{\infty}e^{-x}L_m(x).L_n(x)\,dx=\delta_{mn}$$