Hermite Differential Equation and Hermite Polynomials

The differential equation

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0 \qquad \text{............(1)}$$

Where n is a constant, is called *Hermite Differential Equation* . The series solution of Hermite Equation (1) may be expressed as

$$y = \sum_{r=0}^{\infty} a_r x^{k+r} \qquad \text{.............(2)}$$

$$\therefore \quad \frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (k+r) x^{k+r-1} \qquad \text{.............(3)}$$

and $$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-2} \qquad \text{............. (4)}$$

Substituting these values in equation (1); we get

$$\sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-2} - 2x\sum_{r=0}^{\infty} a_r (k+r) x^{k+r-1} + 2n\sum_{r=0}^{\infty} a_r x^{k+r} = 0$$

$$or \quad \sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-2} - 2\sum_{r=0}^{\infty} a_r (k+r-n) x^{k+r} = 0 \qquad \text{..........(5)}$$

Equating to the coefficients of lowest power of x by putting r=0(i.e. Coefficient of x^{k-2}), we get

$a_0 k(k-1) = 0$ ....(6)

As $a_0 \neq 0$ being coefficient of first term; therefore we must have either

$$k=0 \quad \text{or} \quad k=1 \qquad \text{.............................. (7)}$$

Now equating to zero the coefficient of x^{k-1} in (5), we get

$$a_1(k+1)k = 0 \qquad \text{……………….. (8)}$$

Since $k+1 \neq 0$ for any value of k given by (7); therefore equation (8) implies either that

$$k = 0 \quad \text{or} \quad a_1 = 0 \quad \text{or both are zero} \quad \text{………. (9)}$$

Now equating to zero the coefficient of general term x^{k+r}; we get

$$a_{r+2}(k+r+2)(k+r+1) - 2a_r(k+r-n) = 0$$

or $$a_{r+2} = \frac{2(k+r-n)}{(k+r+2)(k+r+1)} a_r$$

or $$a_{r+2} = \frac{2(k+r)-2n}{(k+r+2)(k+r+1)} a_r \qquad \text{………………… (10)}$$

Now there arises two cases :

Case (i) when k = 0, we have from (10),

$$a_{r+2} = \frac{2r-2n}{(r+2)(r+1)} a_r \qquad \text{……………………(11)}$$

Substituting r = 0,2,4 …….. etc., we get

For r=0 $\quad a_2 = \frac{-2n}{2.1} . a_0 = \frac{-2n}{2!} a_0$

For r=2 $\quad a_4 = \frac{4-2n}{4.3} a_2 = \frac{4-2n}{4.3} . \left(\frac{-2n}{2.1} . a_0\right) = \frac{(-2)^2 n(n-2)}{4!} a_0$

For r=4 $a_6 = \frac{8-2n}{6.5} a_4 = \frac{8-2n}{6.5} \frac{(-2)^2 n(n-2)}{4!} a_0 = \frac{(-2)^3 n(n-2)(n-4)}{6!} a_0$

And so on. In general

$$a_{2m} = \frac{(-2)^m n(n-2) \,\ldots\ldots\, (n-2m+2)}{2m!} a_0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

Again substituting r = 1,3,5 …… etc in (10); we get

For r=1 $a_3 = \frac{2-2n}{3.2} a_1 = -\frac{2(n-1)}{3!} a_1$

$$a_{r+2} = \frac{2r-2n}{(r+2)(r+1)} a_r$$

For r=3 $a_5 = \frac{6-2n}{5.4} a_3 = \frac{6-2n}{5.4} . [-\frac{2(n-1)}{3!} a_1]$

$= (-2)^2 \frac{(n-1)(n-3)}{5!} a_1$

And so on In general

$$a_{2m+1} = (-2)^m \frac{(n-1)(n-3) \,\ldots\ldots\, (n-2m+1)}{(2m+1)!} a_1$$

Now, if $a_1 \neq 0$; then we have

$$y = \sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \ldots..$$

$$= a_0 [1 - \frac{2n}{2!} x^2 + \frac{(-2)^2 n(n-2)}{4!} x^4 \text{--------} + (-2)^m \frac{n(n-2) \ldots\ldots (n-2m+2)}{2m!} x^{2m} + \ldots..]$$

$$+ a_1 [x - \frac{2(n-1)}{3!} x^3 + \frac{(-2)^2 (n-1)(n-3)}{5!} x^5 + \ldots\ldots$$

$$+ (-2)^m \frac{(n-1)(n-3) \ldots\ldots (n-2m+1)}{(2m+1)!} x^{2m+1} + \ldots\ldots] \quad \ldots\ldots\ldots\ldots\ldots (13)$$

In case when $a_1 = 0$, equation (13) reduces to

$$y = a_0 [1 - \frac{2n}{2!} x^2 + \frac{2^2 n(n-2)}{4!} x^4 + \text{-----} + (-2)^m \frac{n(n-2) \ldots\ldots (n-2m+2)}{2m!} x^{2m} + \ldots..]$$

$= y_1$ (assume) ……..……….(14)

Case (ii) : when k = 1, we have from (10),

$$a_{r+2} = \frac{2(k+r) - 2n}{(k+r+2)\,(k+r+1)} a_r$$

$$a_{r+2} = \frac{2(1+r) - 2n}{(r+3)\,(r+2)} a_r \qquad \ldots\ldots\ldots\ldots\ldots . \qquad (15)$$

Substituting r = 1,3 ,5 ………. We get

$a_3 = a_5 = a_7 = \ldots\ldots.. = 0$ (each)

Since in this case a_1 must be zero (refer equation 8)

Substituting r = 0,2,4….. etc. in equation(15), we get

For r=0 $a_2 = \frac{2-2n}{3.2} a_0 = \frac{-2(n-1)}{3!} a_0$

$$a_{r+2} = \frac{2(1+r) - 2n}{(r+3)\,(r+2)} a_r$$

For r=2 $a_4 = \frac{6-2n}{5.4}a_2 = \frac{6-2n}{5.4}.\{-\frac{2(n-1)}{3!}a_0\} = \frac{2\ (n-1)(n-3)}{5!}a_0$

And so on

In general $a_{2m} = (-2)^m \frac{(n-1)(n-3)\ ......\ (n-2m+1)}{(2m+1)\,!}a_0$

Hence $y = \sum_r a_r x^{r+1} = a_0x+a_1x^2+a_2x^3+a_3x^4+a_4x^5 \ ..... + ...$ (Since $a_1=a_3=a_5=...=0$)

$= a_0 \left[x - \frac{2(n-1)}{3!} x^3 + \frac{(-2)^2(n-1)(n-3)}{5!}x^5 + --- (-2)^m \frac{(n-1)(n-3)\ ......\ (n-2m+1)}{(2m+1)\,!}x^{2m+1} + ...\right]$

$= y_2$ (say) ....(16)

Inspection of equations (13) and (16) shows that (16) is a part of equation as given by (13). The two are solutions of the same equation. So we can say that (16) is not a part of (13); but it is a separate solution. Hence $a_1 = 0$ and so the solution in case k=0 must be given by (14). In view of this, General Solution of Hermite Equation is given by

$$y = Ay_1 + By_2$$

Where A and B are arbitrary constants and y_1, y_2 are given by equations (14) and (16)

Hermite Polynomials : Let us now investigate the general solution for n to be even or odd

When *n* is an even integer and $a_0 = (-1)^{n/2} \frac{n\,!}{\{(n/2)\,!\}}$; then in equation (14) the terms containing x^n is

'm' term in equation 14 is $(-\ 2)^m \frac{n(n-2).......\ (n-2m+2)}{2m\,!}x^{2m}$ (2m=n then m=n/2) hence the term become as $(-\ 2)^{n/2} \frac{n(n-2)\ ......(n-n+2)}{n!}x^n$

$$= (-\ 1)^{n/2}\frac{n\,!}{\{(n/2)\,!\}}(-\ 2)^{n/2} \frac{n(n-2)\ ......(n-n+2)}{n!}x^n$$

$$= (2)^{n/2}(2)^{n/2}\frac{\frac{n}{2}.(\frac{n}{2}-1)(\frac{n}{2}-2)---1}{\{(n/2)\,!\}}x^n = (2x)^n$$

'm-1' term in equation 14 is $(-\ 2)^{m-1} \frac{n(n-2).......\ (n-2(m-1)+2)}{2(m-1)\,!}x^{2(m-1)} \Rightarrow$

$(-\ 2)^{m-1} \frac{n(n-2).......\ (n-2m+4)}{(2m-2)\,!}x^{(2m-2)}$(2m=n then m=n/2) hence the term become as

$(-\ 2)^{\frac{n}{2}-1} \frac{n(n-2)\ ......(n-n+4)}{(n-2)!}x^{n-2}$

$$(-1)^{n/2}\frac{n!}{\{(n/2)!\}}(-2)^{\frac{n}{2}-1}\quad\frac{n(n-2)\ldots\ldots(n-n+4)}{(n-2)!}x^{n-2}$$

$$\frac{n!}{\{(n/2)!\}}(-2)^{\frac{n}{2}-1}(-2)^{\frac{n}{2}-1}\frac{\frac{n}{2}.(\frac{n}{2}-1)(\frac{n}{2}-2)---1)}{(n-2)!}x^{n-2}.\ \frac{n(n-1)}{n(n-1)}$$

$$(2)^{n-2}\ \frac{n(n-1)}{1!}x^{n-2}=\frac{n(n-1)}{1!}(2x)^{n-2}$$

'm-2' term in equation 14 is $(-2)^{m-2}\frac{n(n-2)\ldots\ldots(n-2(m-2)+2)}{2(m-2)!}x^{2(m-2)}\Rightarrow$

$(-2)^{m-2}\frac{n(n-2)\ldots\ldots(n-2m+6)}{(2m-4)!}x^{(2m-4)}$(2m=n then m=n/2) hence the term become as

$$(-2)^{\frac{n}{2}-2}\quad\frac{n(n-2)\ldots\ldots(n-n+6)}{(n-4)!}x^{n-4}$$

$$(-1)^{n/2}\frac{n!}{\{(n/2)!\}}(-2)^{\frac{n}{2}-2}\quad\frac{n(n-2)\ldots\ldots(n-n+6)}{(n-4)!}x^{n-4}$$

$$\frac{n!}{\{(n/2)!\}}(-2)^{\frac{n}{2}-2}(-2)^{\frac{n}{2}-2}\frac{\frac{n}{2}.(\frac{n}{2}-1)(\frac{n}{2}-2)---3)}{(n-4)!}x^{n-4}.\ \frac{n(n-1)(n-2)(n-3).2}{n(n-1)(n-2)(n-3).2}$$

$$=(2)^{n-4}\ \frac{n(n-1)(n-2)(n-3)}{2!}x^{n-4}=\frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4}$$

Similarly, the term containing x^{n-1} is $\frac{-n(n-1)}{1!}(2x)^n$ and term containing x^{n-4} is $\frac{n\,(n-1)\,(n-2)\,(n-3)}{2!}(2x)^{n-4}$ and so on. Obviously, the series terminates at the n^{th} term. We have

$y = (2x)^n - \frac{n(n-1)}{1!}(2x)^{n-2} + \frac{n\,(n-1)\,(n-2)\,(n-3)}{2!}(2x)^{n-4} + \ldots\ldots + (-1)^{n/2}\frac{n!}{\{(n/2)!\}}$.....(18)

When *n* is and odd integer and $a_0 = (-1)^{(n+1)/2}\frac{(n+1)!}{\{(n+1)/2)!\}}$; in equation (16), the term containing x^n is

'm' term in equation 16 is $(-2)^m\frac{(n-1)(n-3)\ldots\ldots(n-2m+1)}{(2m+1)!}x^{2m+1}$ [2m+1=n then m=(n-1)/2] hence the term become as

$$(-2)^{(n-1)/2}\frac{(n-1)(n-3)\ldots\ldots(n-2[\frac{(n-1)}{2}]+1)}{n!}x^n=(-2)^{(n-1)/2}\frac{(n-1)(n-3)\ldots\ldots 2}{n!}x^n$$

$$= (-1)^{(n+1)/2} \frac{(n+1)!}{\{(n+1)/2)!\}} (-2)^{(n-1)/2} \frac{(n-1)(n-3) \ldots\ldots 2}{n!} x^n$$

$$= (2)^{(n-1)/2} \frac{(n+1)(n-1)(n-3)\ldots 4 . 2}{\{(n+1)/2\}!} x^n$$

$$= (2)^{(n-1)/2} (2)^{(n+1)/2} \frac{\frac{(n+1)}{2}\frac{(n-1)}{2}\frac{(n-3)}{2}\ldots 2.1}{\{(n+1)/2\}!} x^n$$ = $(2x)^n$ and so on.

'm-1' term in equation 16 is $(-2)^{m-1} \frac{(n-1)(n-3) \ldots\ldots [n-2(m-1)+1]}{(2(m-1)+1)!} x^{2(m-1)+1}$

$= (-2)^{m-1} \frac{(n-1)(n-3) \ldots\ldots [n-2m+3]}{(2m-1)!} x^{2m-1}$ [2m+1=n then m=(n-1)/2] hence the term

become as $(-2)^{\{[(n-1)/2]-1\}} \frac{(n-1)(n-3) \ldots\ldots (n-2[\frac{(n-1)}{2}]+3)}{(n-2)!} x^{2(\frac{(n-1)}{2})-1}$

$$= (-2)^{(n-3)/2} \frac{(n-1)(n-3) \ldots\ldots 4}{(n-2)!} x^{n-2}$$

$$(-1)^{(n+1)/2} \frac{(n+1)!}{\{(n+1)/2)!\}} (-2)^{(n-3)/2} \frac{(n-1)(n-3) \ldots\ldots 4}{(n-2)!} x^{n-2}$$

$$= (-2)^{(n-3)/2} \frac{(n+1)!}{\{(n+1)/2)!\}} \frac{(n-1)(n-3) \ldots\ldots 4}{(n-2)!} x^{n-2} \frac{n(n-1)}{n(n-1)}$$

$$= (-2)^{(n-3)/2} \frac{(n+1)!}{\{(n+1)/2)!\}} \frac{(n-1)(n-3) \ldots\ldots 4}{(n-2)!} x^{n-2} \frac{n(n-1)}{n(n-1)}$$

$$= (2)^{(n-3)/2} (2)^{(n-1)/2} \frac{\frac{(n+1)}{2}\frac{(n-1)}{2}\frac{(n-3)}{2}\ldots 2.1}{\{(n+1)/2\}!} x^{n-2} \frac{n(n-1)}{1} = \frac{n(n-1)}{1!} (2x)^{n-2}$$

The last term will be $(-1)^{(n+1)/2} \frac{(n+1)!}{\{(n+1)/2)!\}} x$; therefore, for odd *n*

$$y = [(2x)^n - \frac{n(n-1)}{1!}(2x)^{n-2} + \ldots + (-1)^r \frac{n(n-1)(n-3) \ldots (n-2r+1)}{r!} (2x)^{n-2r}$$

$$----+(-1)^{(n+1)/2} \frac{(n+1)!}{\{(n+1)/2)!\}}] \quad \ldots\ldots \ldots(19)$$

The values of *y* in equations (18) and (19) are called *Hermite Polynomials* of degree *n* for even and odd integers respectively and are denoted by $H_n(x)$

Thus we have a Hermite *Polynomial* of degree n, for *n* being a positive integer.

$$H_n(x) = \sum_{r=0}^{p} (-1)^r \frac{n!}{r!\,(n-2r)!} (2x)^{n-2r} \quad \ldots\ldots\ldots\ldots\ldots\ldots(20)$$

Where p = $\begin{cases} n/2 & if\ n\ is\ even \\ (n-1)/2 & if\ n\ is\ odd \end{cases}$

We observe that

$H_n(0) = (-1)^n \frac{n!}{\{(n/2)!\}}$ if n is an even integer

$H_n(0) = 0$ if n is an odd integer

$$for\ even\ H_n(x) = \sum_{r=0}^{n/2} (-1)^r \frac{n!}{r!\,(n-2r)!} (2x)^{n-2r} \Rightarrow H_n(0) = (-1)^{n/2} \frac{n!}{(n/2)!\,[n-(n/2)]!} (2x)^{n-2(n/2)}$$

$$H_n(0) = (-1)^{n/2} \frac{n!}{(n/2)!\,[n-2(n/2)]!} (2x)^{n-2(n/2)} \Rightarrow H_n(0) = \sum_{r=0}^{n/2} (-1)^{n/2} \frac{n!}{(n/2)!\,0!} (2x)^0$$

$$H_n(0) = (-1)^{n/2} \frac{n!}{(n/2)!}$$

$$for\ odd\quad H_n(x) = \sum_{r=0}^{(n-1)/2} (-1)^r \frac{n!}{r!\,(n-2r)!} (2x)^{n-2r}$$

$$H_n(x) = (-1)^{(n-1)/2} \frac{n!}{[(n-1)/2]!\,\{[n-2(n-1)/2]\}!} (2x)^{n-2[(n-1)/2]}$$

$$H_n(x) = (-1)^{(n-1)/2} \frac{n!}{[(n-1)/2]!\,[1]!} (2x)^1 \Rightarrow H_n(0) = 0$$

From expression (20), we can write values of Hermite polynomials of different orders, even or odd. They are as follows

$$H_n(x) = \sum_{r=0}^{p} (-1)^r \frac{n!}{r!\,(n-2r)!} (2x)^{n-2r}$$

$$H_0(x) = \sum_{r=0}^{p} (-1)^r \frac{n!}{r!\,(n-2r)!} (2x)^{n-2r}$$

$n = 0 \Rightarrow r = n/2 = 0$

$$H_0(x) = \sum_{r=0}^{0} (-1)^0 \frac{0!}{0!\,(0)!} (2x)^0 = 1$$

$$H_1(x) = \sum_{r=0}^{0} (-1)^0 \frac{1!}{0!\,(1)!} (2x)^1 = 2x$$

$for\ odd\ n = 1 \Rightarrow r = (n-1)/2 = 0$

$$H_2(x) = \sum_{r=0}^{1} (-1)^r \frac{2!}{r!\,(2-2r)!} (2x)^{2-2r}$$

$$= (-1)^0 \frac{2!}{0!\,(2-0)!} (2x)^{2-0} + (-1)^1 \frac{2!}{1!\,(2-2)!} (2x)^{2-2} = 4x^2 - 2$$

$$H_3(x) = \sum_{r=0}^{1} (-1)^r \frac{3!}{r!\,(3-2r)!} (2x)^{3-2r}$$

for odd $n = 3 \Rightarrow r = (n-1)/2 = 1$

$$= (-1)^0 \frac{3!}{0!\,(3-0)!} (2x)^{3-0} + (-1)^1 \frac{3!}{1!\,(3-2)!} (2x)^{3-2} = 8x^3 - 12x$$

$$H_4(x) = \sum_{r=0}^{2} (-1)^r \frac{4!}{r!\,(4-2r)!} (2x)^{4-2r}$$

for even $n = 4 \Rightarrow r = n/2 = 2$

$$= (-1)^0 \frac{4!}{0!\,(4-0)!} (2x)^{4-0} + (-1)^1 \frac{4!}{1!\,(4-2)!} (2x)^{4-2} + (-1)^2 \frac{4!}{2!\,(4-4)!} (2x)^{4-4}$$

$$= 16x^4 - 48x^2 + 12$$