$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ------$$

Generating function of Hermite polynomials

Theorem. The function e^{2zx-z^2} is called the generating function of Hermite polynomials

I.e., $f(x,z) = e^{2zx-z^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n$

Proof. We have

$$e^{2zx-z^2} = e^{2zx}\ e^{-z^2}$$

$$= \sum_{r=0}^{\infty} \frac{(2zx)^r}{r!} . \sum_{s=0}^{\infty} \frac{(-z^2)^s}{s!}$$

$$= \sum_{r,s=0}^{\infty} (-1)^s \frac{(2x)^r}{r!\,s!} . z^{r+2s}$$

The coefficient of z^n (for fixed value of s) on R.H.S is obtained by putting r+2s = n i.e., r = n-2s and is given by

$$(-1)^s \frac{(2x)^{n-2s}}{(n-2s)!\,s!}$$

The total coefficient of z^n is obtained by summing over all allowed values of s and since r = n-2s

$$n-2s \geq 0 \text{ or } s \leq \frac{n}{2}$$

Thus if *n* is even; s goes from 0 to $\frac{n}{2}$ and if *n* is odd, s goes from 0 to $\frac{n-1}{2}$

∴ Coefficient of z^n = $\sum_{s=0}^{n/2} (-1)^s \frac{(2x)^{n-2s}}{(n-2s)!\,s!} = \frac{H_n(x)}{n!}$

$$H_n(x) = \sum_{r=0}^{p} (-1)^r \frac{n!}{r!\,(n-2r)!} (2x)^{n-2r}$$

Hence we may write

$$e^{2zx-z^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n \quad \text{……………(1)}$$

Thus the function $e^{2zx-z^2} = e^{\{x^2-(z-x)^2\}}$ generates all the Hermite polynomials and hence it is called the **generating function of Hermite polynomials.**