Formulas

1. $e^{-\infty} = 0$
2. $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$
3. $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$
4. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ------$

Orthogonality of Hermite Polynomials

The orthogonality property of Hermite Polynomials is

$$\int_{-\infty}^{+\infty} e^{-x^2} H_n(x)\, H_m(x)\, dx = 2^n\, n!\, \sqrt{\pi}\ \ \delta_{mn}$$

Where δ_{mn} is *Kronecker delta symbol* such that $\delta_{mn} = 1$ for m=n and 0 for m≠n.

Proof : Since $H_n(x)$ is the solution of Hermite Equation of order n ; we have

$$H''_n(x) - 2x\, H'_n(x) + 2n\, H_n(x) = 0 \quad \ldots\ldots\ldots\ldots\ldots (1)$$

Multiplying throughout by e^{-x^2}; we get

$$e^{-x^2} H''_n(x) - 2x\, e^{-x^2} H'_n(x) + 2n\, e^{-x^2} H_n(x) = 0$$

or $$\frac{d}{dx}[e^{-x^2} H'_n(x)] + 2n\, e^{-x^2} H_n(x) = 0 \quad \ldots\ldots\ldots\ldots (2)$$

Similarly, we have

$$\frac{d}{dx}[e^{-x^2} H'_m(x)] + 2m\, e^{-x^2} H_m(x) = 0 \quad \ldots\ldots\ldots\ldots (3)$$

Multiplying equation (2) by $H_m(x)$ and (3) by $H_n(x)$ and subtracting, we get

$$2(n-m)\, e^{-x^2} H_n(x)\, H_m(x) = H_n(x) \frac{d}{dx}[e^{-x^2} H'_m(x)] - H_m(x) \frac{d}{dx}[e^{-x^2} H'_n(x)]$$

$$= \frac{d}{dx}[e^{-x^2} \{ H_n(x)\, H'_m(x) - H'_n(x)\, H_m(x)\}] \quad \ldots\ldots\ldots\ldots\ldots (4)$$

$$\frac{d}{dx}[\, e^{-x^2} \{ H_n(x)\ H'_m(x) - H'_n(x)\, H_m(x)\} \,]$$

$$= H_n(x)\frac{d}{dx}[e^{-x^2}H'_m(x)] + H'_n(x)e^{-x^2}H'_m(x) - H_m(x)\frac{d}{dx}[e^{-x^2}H'_n(x)] - H'_m(x)e^{-x^2}H'_n(x)$$

Integrating above equation with respect to x between the limits $-\infty$ to $+\infty$; we get

$$2(n-m)\int_{-\infty}^{+\infty} e^{-x^2} H_n(x)\, H_m(x)\, dx$$

$$= [\, e^{-x^2} \{ H_n(x)\ H_m(x) - H_n(x)\, H_m(x)\}]_{-\infty}^{+\infty} \qquad \ldots\ldots\ldots\ldots(5)$$

$$= 0 \text{ since } e^{-x^2} \to 0 \text{ as } x = \pm\ \infty$$

Thus for $m \neq n$, we have $\int_{-\infty}^{+\infty} e^{-x^2} H_n(x)\, H_m(x)\, dx = 0 \qquad \ldots\ldots\ldots\ldots\ (6)$

From the generating function of Hermite polynomials, we have

$$e^{2xz - z^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n \qquad \ldots\ldots\ldots\ldots\ (7)$$

And

$$e^{2xt - t^2} = \sum_{m=0}^{\infty} \frac{H_m(x)}{m!} t^m \qquad \ldots\ldots\ldots\ldots\ (8)$$

Multiplying (7) and (8) together and then by e^{-x^2}; we get

$$e^{-x^2} e^{2xt - t^2} . e^{2xz - z^2} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} e^{-x^2} \frac{H_n(x)\, H_m(x)}{n!\, m!} z^n t^m \qquad \ldots\ldots\ldots\ldots\ (9)$$

Integrating equation (9) over x from $-\infty$ to $+\infty$; we get

$$\int_{-\infty}^{+\infty} e^{-x^2} e^{2xt - t^2} . e^{2xz - z^2}\, dx$$

$$= \sum_{n=0}^{\infty} z^n t^n \int_{-\infty}^{+\infty} e^{-x^2} \frac{[H_n(x)]^2}{(n!)^2} dx + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} z^n t^m \int_{-\infty}^{+\infty} e^{-x^2} \frac{[H_n(x)\, H_m(x)]}{m!n!} dx$$

The last term on right-hand value vanishes by virtue of orthogonal property (6) and therefore we have

$$\sum_{n=0}^{\infty} \frac{(zt)^n}{(n!)^2} \int_{-\infty}^{+\infty} e^{-x^2} [H_n(x)]^2 dx = \int_{-\infty}^{+\infty} e^{2xz - z^2 + 2xt - t^2 - x^2} dx$$

$$= \int_{-\infty}^{+\infty} e^{-(x-z-t)^2} e^{2zt} dx \quad \{\because (a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca\}$$

$$= e^{2zt} \sqrt{\pi} \qquad (\because \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi})$$

$x - z - t = y \Rightarrow dx = dy \quad then \int_{-\infty}^{+\infty} e^{-y^2} dy = \sqrt{\pi}$

$$= \sqrt{\pi} \sum_{n=0}^{\infty} \frac{2^n (zt)^n}{n!} \qquad \text{………… (10)}$$

Comparing coefficients $(zt)^n$ on either side , we get

$$\frac{1}{(n!)^2} \int_{-\infty}^{+\infty} e^{-x^2} [H_n(x)]^2 dx = \frac{\sqrt{\pi}\, 2^n}{n!}$$

or $$\int_{-\infty}^{+\infty} e^{-x^2} [H_n(x)]^2 dx = 2^n n! \sqrt{\pi} \qquad \text{………………. (11)}$$

Combining (6) and (11) we get

$$\int_{-\infty}^{+\infty} e^{-x^2} H_n(x) H_m(x) dx = 2^n n! \sqrt{\pi} \text{ …………… (12)}$$

The *Hermite* polynomials are specifically useful in analysing the quantum mechanical problem of *Harmonic oscillator*.