6.31. Recurrence formulae for Hermite polynomials

(i) $H'_n(x) = 2n\,H_{n-1}(x)$

Proof. We have $\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n = e^{2zx - z^2}$

Differentiating above equation with respect to x ; we get

$$\sum_{n=0}^{\infty} \frac{H'_n(x)}{n!} z^n = e^{2zx - z^2}.2z$$

$$= 2z \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n = 2 \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^{n+1}$$

$$= 2 \sum_{n=0}^{\infty} \frac{H_{n-1}(x)}{(n-1)!} z^n$$

Equating the coefficients of z^n on both sides, we get

$$\frac{H'_n(x)}{n!} = 2\,\frac{H'_{n-1}(x)}{(n-1)!}$$

Or $H'_n(x) = 2n.\,H_{n-1}(x)$ …….. (1)

(ii) $2x\,H_n(x) = 2n.\,H_{n-1}(x) + H_{n+1}(x)$

Proof : we have $\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n = e^{2zx - z^2}$

Differentiating both sides with respect to z ; we get

$$\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} n z^{n-1} = e^{2zx - z^2}.\,(2x-2z)$$

or $$\sum_{n=0}^{\infty} \frac{H_n(x)}{(n-1)!} z^{n-1} = 2x.e^{2zx - z^2} - 2z.e^{2zx - z^2}$$

or $$\sum_{n=0}^{\infty} \frac{H_n(x)}{(n-1)!} z^{n-1} = 2x \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n - 2z \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n$$

By rearranging we get

$$2x \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n = 2 \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^{n+1} + \sum_{n=1}^{\infty} \frac{H_n(x)}{(n-1)!} z^{n-1}$$

Equating coefficients of z^n on both sides, we get

$$2x \frac{H_n(x)}{n!} = \frac{2H_{n-1}(x)}{(n-1)!} + \frac{H_{n+1}(x)}{n!}$$

$$2x\, H_n(x) = 2n\, H_{n-1}(x) + H_{n+1}(x) \quad \ldots\ldots\ldots\ldots (2)$$

(iii) $H'_n(x) = 2x\, H_n(x) - H_{n+1}(x)$

Proof : Recurrence relation (I) *and* (II) are

$$H'_n(x) = 2n\, H_{n-1}(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots (i)$$

$$2x\, H_n(x) = 2n\, H_{n-1}(x) + H_{n+1}(x) \quad \ldots\ldots\ldots\ldots (ii)$$

Subtracting (ii) from (i) we get

$$H'_n(x) - 2x\, H_n(x) = 2n\, H_{n-1}(x) - 2n\, H_{n-1}(x) - H_{n+1}(x)$$

$$H'_n(x) = 2x\, H_n(x) - H_{n+1}(x)$$

(iv) $H''_n(x) - 2x\, H'_n(x) + 2n\, H_n(x) = 0$

Proof : Hermite Differential Equation is

$$y'' - 2xy' + 2ny = 0$$

As $H_n(x)$ is the solution of above equation, i,e substitution it with y, we get

$$H''_n(x) - 2x\, H'_n(x) + 2n\, H_n(x) = 0 \quad \ldots\ldots\ldots\ldots\ldots (4)$$

Rodrigue's Formula for Hermite Polynomials

From generating function of Hermite Polynomials, we have

$$e^{2zx-z^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n$$

$$e^{x^2 - (z-x)^2} = \frac{H_0(x)}{0!}z^0 + \frac{H_1(x)}{1!}z^1 + \frac{H_2(x)}{2!}z^2 + \ldots + \frac{H_n(x)}{n!}z^n + \frac{H_{n+1}(x)}{(n+1)!}z^{n+1} + \ldots\ldots$$

Differentiating both sides with respect to z, *n* times and then substituting z=0 ; we get

$$\left[\frac{\partial^n}{\partial z^n} e^{x^2 - (z-x)^2}\right]_{z=0} = \frac{H_n(x)}{n!} n!$$

or $$H_n(x) = e^{x^2}\left[\frac{\partial^n}{\partial z^n} e^{-(z-x)^2}\right]_{z=0}$$

Now substituting z-x = t, i,e at z=0, $t = -x$

And $\frac{\partial}{\partial z} = \frac{\partial}{\partial t}$, we get

$$H_n(x) = e^{x^2}\left[\frac{\partial^n}{\partial t^n} e^{-t^2}\right]_{t=-x}$$

$$= e^{x^2}(-1)^n \frac{\partial^n}{\partial x^n}\left(e^{-x^2}\right) \qquad \left(\because t = -x \Rightarrow \frac{\partial}{\partial t} = -\frac{\partial}{\partial x}\right)$$

or $$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Equation (1) represents the **differential form of Hermite Polynomials** and is called the *Rodrigue's formula* for Hermite polynomials.

Substituting n = 0,1,2,3….etc in equation (1) ; we can find the values of Hermite Polynomials. For example,

For n =0, $H_0(x) = (-1)^0 e^{x^2} \frac{d^0}{dx^0}\left(e^{-x^2}\right) = e^{x^2}.e^{-x^2} = 1$

For n =1, $H_1(x) = (-1)^1 e^{x^2} \frac{d}{dx}\left(e^{-x^2}\right) = +2x$

For n =2, $H_2(x) = (-1)^2 e^{x^2} \frac{d^2}{dx^2}(e^{- x^2}) =4x^2 - 2$

$$e^{x^2}\frac{d}{dx}\{e^{-x^2}(- 2x)\} =- e^{x^2}\frac{d}{dx}\{e^{-x^2}(2x)\} = - e^{x^2}\{e^{-x^2}(2) + (2x)e^{-x^2}(- 2x)\}$$

Similarly, $H_3(x) = 8x^3 - 12x$

$H_4(x) = 16x^4 - 48x^2 +12$ and so on