6.47 Representation of various functions in terms of hypergeometric functions

Various familiar functions of mathematical physics are particular cases of hypergeometric function corresponding to suitable choices of parameters α, β, γ and the variable of x.

1. **Legendre's Polynomials :**

(i) from Rodrigue's formula of Legendre Polynomials, we have

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

$$= \frac{1}{n!} \frac{d^n}{dx^n} [(x - 1)^n . \{\frac{1}{2} (x + 1)\}^n]$$

$$= \frac{1}{n!} \frac{d^n}{dx^n} [(x - 1)^n . \{1 - \frac{1}{2} (1 - x) \quad]\}^n$$

$$= \frac{(-1)^n}{n!} \frac{d^n}{dx^n} [(1 - x)^n \{1 - n. \frac{1}{2} (1 - x) + \frac{n(n-1)}{2!} \frac{(1-x)^2}{4} - \frac{n(n-1)(n-2)}{3!} . \frac{(1-x)^3}{8} + ...\}]$$

$$= \frac{(-1)^n}{n!} \frac{d^n}{dx^n} [(1 - x)^n - \frac{n}{2} (1 - x)^{n+1} + \frac{n(n-1)}{2!\, 2^2} (1 - x)^{n+2} - \frac{n(n-1)(n-2)}{3!\, 2^3} (1 - x)^{n+3} + ...]$$

$$= \frac{(-1)^n}{n!} [(- 1)^n n! - \frac{n}{2} (- 1)^n \frac{(n+1)!}{1!} (1 - x) + \frac{n(n-1)}{2!} . \frac{(-1)^n}{2^2} \frac{(n+2)!}{2!} (1 - x)^2 + ...]$$

$$= 1 + \frac{(-n).(n+1)}{1.1!} (\frac{1-x}{2}) + \frac{(-n)(-n+1)(n+1)(n+2)}{2.1.2!} (\frac{1-x}{2})^2 + ...$$

$$\therefore \quad P_n(x) = {}_2F_1(- n, n + 1, 1; \frac{1-x}{2}) \quad .... (1)$$

(ii) substituting $x = cos\, \theta$ in (1); we get

$$P_n(cos\, \theta) = {}_2F_1(- n, n + 1, 1; sin^2 \frac{\theta}{2}) \quad .... (2)$$

Substituting $x = - cos\, \theta$ in (1); we get

$$P_n(- cos\, \theta) = {}_2F_1(- n, n + 1, 1; cos^2 \frac{\theta}{2})$$

or

$(-1)^n P_n(\cos\theta) = {}_2F_1(-n, n+1, 1; \cos^2\frac{\theta}{2})$ $\quad [\because P_n(-x) = (-1)^n P_n(x)]$

$\therefore \quad P_n(\cos\theta) = (-1)^n \; {}_2F_1(-n, n+1, 1; \cos^2\frac{\theta}{2})$ .... (3)

$= (-1)^n \; {}_2F_1(n+1, -n, 1; \cos^2\frac{\theta}{2})$ .... (4)

(since in hypergeometric series $\underline{\alpha}$ and β can be interchanged)

Equations (1), (2), (3) and (4) represent Legendre polynomials in terms of hypergeometric functions.

2. Elementary Function : In hypergeometric function ${}_2F_1(\alpha, \beta, \gamma; x)$ if α *or* β is negative integer; then the series terminates after a finite number of terms and hence changes to polynomial. For example, if $\beta = 0$,

We know $\quad {}_2F_1(\alpha, \beta, \gamma; x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{n! \, (\gamma)_n} x^n = [1 + \frac{\alpha.\beta}{\gamma} x + \frac{\alpha(\alpha+1) \;\; \beta(\beta+1)}{\gamma\,(\gamma+1)2!} x^2 + ...]$

${}_2F_1(\alpha, \beta, \gamma; x) = F(\alpha, 0, \gamma; x) = 1 + \frac{\alpha\beta}{\gamma(\gamma+1)}. x + ... = 1$ .... (5)

From equation (20) of the property of linear transformations of hypergeometric functions; we have

${}_2F_1(\alpha, \beta, \gamma; x) = (1-x)^{-\beta} \; {}_2F_1(\gamma - \alpha, \beta, \gamma; \frac{x}{x-1})$

$\therefore {}_2F_1(\alpha, \gamma - \beta, \gamma; \frac{x}{x-1}) = (1 - \frac{x}{x-1})^{\beta-\gamma} \; {}_2F_1\{\gamma - \alpha, \gamma - \beta, \gamma; \frac{x/(x-1)}{(\frac{x}{x-1} - 1)}\}$

$(1 - \frac{x}{x-1})^{\beta-\gamma} = (\frac{x-1-x}{x-1})^{\beta-\gamma} = (\frac{-1}{x-1})^{\beta-\gamma} = (\frac{1}{1-x})^{\beta-\gamma} = (1-x)^{\gamma-\beta}$

$\frac{x/(x-1)}{(\frac{x}{x-1} - 1)} = \frac{x}{(x-1)} \; \frac{(x-1)}{(x-x+1)} = x$

$= (1-x)^{\gamma-\beta} \; {}_2F_1(\gamma - \alpha, \gamma - \beta, \gamma; x)$ ...... (6)

Hence

${}_2F_1(\alpha, \beta, \gamma; x) = (1-x)^{-\alpha} \; {}_2F_1(\alpha, \gamma - \beta, \gamma; \frac{x}{x-1})$ $\quad$ [*using equation* (19)]

Substituting equation (6) in above equation (19)

${}_2F_1(\alpha, \beta, \gamma; x) = (1-x)^{-\alpha} \; {}_2F_1(\alpha, \gamma - \beta, \gamma; \frac{x}{x-1})$

$= (1-x)^{-\alpha}(1-x)^{\gamma-\beta} \; {}_2F_1(\gamma - \alpha, \gamma - \beta, \gamma; x)$

$$= (1 - x)^{\gamma-\beta-\alpha} \; {}_2F_1(\gamma - \alpha, \gamma - \beta, \gamma; x) \quad .... (7)$$

If $\gamma = \beta$

$${}_2F_1(\alpha, \beta, \beta; x) = (1 - x)^{-\alpha} \; {}_2F_1(\gamma - \alpha, 0, \gamma; x)$$

$$= (1 - x)^{-\alpha}.1 \qquad [using\ equation\ (5)]$$

$$\therefore \qquad (1 - x)^{-\alpha} = {}_2F_1(\alpha, \beta, \beta\,; x) \quad . ... (8)$$

If $\beta = 1$and $\alpha =- n$ then

$$(1 - x)^{n} = {}_2F_1(- n, 1, 1\,; x) \quad .... (9)$$

Replacing $x$by $(1 - x)$;we get

$$x^{n} = {}_2F_1(- n, 1, 1\,; x) \quad .... (10)$$

Eq (9) and (10) represent algebraic functions like $(1 - x)^{n}$ $and\ x^{n}$in terms of *hypergeometric functions.*