Gauss Hypergeometric Equation

The differential equation

$$x(1 - x)\frac{d^2y}{dx^2} + [\gamma - (\alpha + \beta + 1)x]\frac{dy}{dx} - \alpha\beta y = 0 \quad \ldots\ldots\ldots\ldots\ldots (1)$$

With α, β, γ as constants is known as Gauss Hypergeometric Equation or simply the Gauss Equation or Hypergeometric equation. The solution hypergeometric equation is called the hypergeometric function.

Dividing the hypergeometric equation throughout by $x^2 - x$;we get

$$\frac{d^2y}{dx^2} + X_1\frac{dy}{dx} + X_2 y = 0. \quad \ldots\ldots\ldots\ldots (2)$$

Where $X_1 = \frac{(\alpha+\beta+1)x-\gamma}{x^2-x}$ and $X_2 = \frac{\alpha\beta}{x^2-x}$

It is obvious that $X_1 \to \infty$for x=0 or 1 or ∞

$X_2 \to \infty$for x=0 or 1

Therefore , x=0,1 and ∞ are singularities of equation(1).

Thus we can integrate(1) in series about x=0 or 1 or ∞. We therefore discuss the series integration in three cases.

Case (a) when x=0; then taking the series solution (1) as

$$y = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} \quad \ldots\ldots\ldots\ldots\ldots (3)$$

So that

$$\frac{dy}{dx} = \sum_{\lambda=0}^{\infty} a_\lambda (k + \lambda)x^{k+\lambda-1}$$

And

$$\frac{d^2y}{dx^2} = \sum_{\lambda=0}^{\infty} a_\lambda (k + \lambda)(k + \lambda - 1)x^{k+\lambda-2}$$

Substituting these values in (1) ; we get

$$x(1 - x)\sum_\lambda a_\lambda (k + \lambda)(k + \lambda - 1)x^{k+\lambda-2} + [\gamma - (\alpha + \beta + 1)x]\sum_\lambda a_\lambda (k + \lambda)x^{k+\lambda-1} - \alpha\beta\sum_\lambda a_\lambda x^{k+\lambda} = 0$$

$$\sum_\lambda a_\lambda (k + \lambda)(k + \lambda - 1)x^{k+\lambda-1} - \sum_\lambda a_\lambda (k + \lambda)(k + \lambda - 1)x^{k+\lambda} + \gamma\sum_\lambda a_\lambda (k + \lambda)x^{k+\lambda-1}$$

$$- (\alpha + \beta + 1)\sum_{\lambda} a_{\lambda}(k + \lambda)x^{k+\lambda} - \alpha\beta\sum_{\lambda} a_{\lambda}x^{k+\lambda} = 0$$

$$\sum_{\lambda} a_{\lambda}[(k + \lambda)(k + \lambda - 1) + (\alpha + \beta + 1)(k + \lambda) + \alpha\beta]x^{k+\lambda}$$

$$- \sum_{\lambda} a_{\lambda}[(k + \lambda)(k + \lambda - 1) - \gamma(k + \lambda)]x^{k+\lambda-1} = 0$$

$$\sum_{\lambda} a_{\lambda}[(k + \lambda)^2 - (k + \lambda) + (\alpha + \beta)(k + \lambda) + (k + \lambda) + \alpha\beta]x^{k+\lambda}$$

$$- \sum_{\lambda} a_{\lambda}[(k + \lambda)(k + \lambda - 1) - \gamma(k + \lambda)]x^{k+\lambda-1} = 0$$

or

$$\sum_{\lambda} a_{\lambda}[\{(k + \lambda)^2 + (\alpha + \beta)(k + \lambda) + \alpha\beta\}x^{k+\lambda} - (k + \lambda)(k + \lambda + \gamma - 1)x^{k+\lambda-1}] = 0 \quad \text{..........} (4)$$

As equation(4) is an identity, the coefficients of various powers of x must be equal to zero. Equating to zero the coefficient of lowest power of x i,e x^{k-1};(putting λ =0) we get

$$a_0 k(k + \gamma - 1) = 0 \quad \text{.........} (5)$$

This equation is called the indicial *equation*. As $a_0 \neq 0$, being the coefficient term of the series; we must have

$$k = 0 \qquad \text{and} \qquad k = 1 - \gamma \quad \text{.........} (6)$$

Equating to zero the coefficient of next higher power of x i.e x^k ; we get

$$a_0[k^2 + (\alpha + \beta)k + \alpha\beta] - (k + 1)(k + \gamma)a_1 = 0$$

$$a_1 = \frac{k^2+(\alpha+\beta)k+\alpha\beta}{(k+1)(k+\gamma)}\, a_0 = \frac{(k+\alpha)(k+\beta)}{(k+1)(k+\gamma)}\, a_0 \quad \text{.........} (7)$$

Again equating to zero the coefficient of general term, i.e $x^{k+\lambda}$, we get recurrence relation between the coefficients $a_{\lambda}'s$

$$a_{\lambda}\{(k + \lambda)^2 + (\alpha + \beta)(k + \lambda) + \alpha\beta\} - (k + \lambda + 1)(k + \lambda + \gamma)a_{\lambda+1} = 0$$

$$a_{\lambda+1} = \frac{(k+\lambda+\alpha)(k+\lambda+\beta)}{(k+\lambda+1)(k+\lambda+\gamma)}\, a_{\lambda} \quad \text{.............} (8)$$

$$(k + \lambda)^2 + (\alpha + \beta)(k + \lambda) + \alpha\beta = (k + \lambda)^2 + \alpha(k + \lambda) + \beta(k + \lambda) + \alpha\beta$$
$$= (k + \lambda)(k + \lambda + \alpha) + \beta(k + \lambda + \alpha) = (k + \lambda + \alpha)(k + \lambda + \beta)$$

Choice (1) when k=0, we have from (8)

$$a_{\lambda+1} = \frac{(\lambda+\alpha)(\lambda+\beta)}{(\lambda+1)(\lambda+\gamma)}\, a_\lambda \qquad \text{..........} (9)$$

Substituting $\lambda = 0, 1, 2.....;$ *we get*

$\lambda = 0$ $\qquad a_1 = \frac{\alpha.\beta}{1.\gamma}\, a_0$

$\lambda = 1$ $\qquad a_2 = \frac{(1+\alpha)(1+\beta)}{2(1+\gamma)}\, a_1 = \frac{\alpha(1+\alpha)\,\beta(1+\beta)}{2!\gamma(1+\gamma)}\, a_0$

$\lambda = 2$ $\qquad a_3 = \frac{(2+\alpha)(2+\beta)}{3(2+\gamma)}\, a_2 = \frac{\alpha(\alpha+1)\,(\alpha+2).\beta(\beta+1)(\beta+2)}{3!\gamma(\gamma+1)(\gamma+2)}\, a_0$ and so on

Substituting k=0 and the values of $a_1, a_2, a_3......$ *etc*in (3) and using the notation $(\alpha)_n = \alpha(\alpha + 1)(\alpha + 2).....(\alpha + n - 1)$we get

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$$

$$= a_0[1 + \frac{\alpha.\beta}{1.\gamma}\, x + \frac{\alpha(1+\alpha)\,\beta(1+\beta)}{2!\gamma(1+\gamma)}\, x^2 + \frac{\alpha(\alpha+1)\,(\alpha+2).\beta(\beta+1)(\beta+2)}{3!\gamma(\gamma+1)(\gamma+2)} x^3 + ... + \frac{(\alpha)_n(\beta)_n}{n!\,(\gamma)_n} x^n + ...]$$

$$= a_0 \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{n!\,(\gamma)_n} x^n$$

$$= a_0 \sum_{n=0}^{\infty} \frac{\alpha(\alpha+1)........(\alpha+n-1)\,.\beta(\beta+1).....(\beta+n-1)}{n!\gamma(\gamma+1)....(\gamma+n-1)}\, x^n \qquad \text{..........} (10a)$$

If $a_0 = 1,$ then

$$y = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{n!\,(\gamma)_n} x^n \qquad \text{............} (10b)$$

Is a particular solution of given equation and the series is known as *hypergeometric series*

Convergence of the series. The m^{th} term of the series

$$y_m = \frac{(\alpha)_m(\beta)_m}{m!\,(\gamma)_m} x^m$$

And the $(m + 1)^{th}$term is

$$y_{m+1} = \frac{(\alpha)_{m+1}(\beta)_{m+1}}{(m+1)!\,(\gamma)_{m+1}}\, x^{m+1}$$

So that

$$\frac{y_{m+1}}{y_m} = \frac{(\alpha)_{m+1}(\beta)_{m+1}}{(m+1)!\,(\gamma)_{m+1}}\; \frac{m!(\gamma)_m}{(\alpha)_m(\beta)_m}.\frac{x^{m+1}}{x^m}$$

$(\alpha)_m = \alpha(\alpha+1)(\alpha+2).....(\alpha+m-1)$ $(\alpha)_{m+1} = \alpha(\alpha+1)(\alpha+2).....(\alpha+m-1)(\alpha+m)$

$$= \frac{(\alpha+m)(\beta+m)}{(m+1)(\gamma+m)} x$$

$$\therefore \quad \lim_{m\to\infty} \left|\frac{y_{m+1}}{y_m}\right| = \lim_{m\to\infty} \frac{(1+\frac{\alpha}{m})(1+\frac{\beta}{m})}{(1+\frac{1}{m})(1+\frac{\gamma}{m})}|x| = |x|$$

The series is convergent if $\lim_{m\to\infty}\left|\frac{y_{m+1}}{y_m}\right| < 1$or $|x| < 1$. Obviously, the radius of convergency is unity.

The series $y = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{n!\,(\gamma)_n}$for x<1 is hypergeometric series and its solution is called hypergeometric function, denoted by ${}_2F_1(\alpha, \beta, \gamma;\ x)$where the leading subscript 2 indicates that the first two symbols in bracket $(\alpha\ and\ \beta)$appear in the numerator and the second subscript 1 indicates that one symbol (γ)appears in the denominator or $F(\frac{\alpha\beta}{\gamma}. x)$Thus, the solution for the *hypergeometric equation* for k=0 is

$$y = a\,{}_2F_1(\alpha, \beta, \gamma;\ x) \qquad .... (11)$$

choice(ii): when $k = 1-\gamma$;the solution of hypergeometric differential equation for this value of k is given by

$$y = \sum_{\lambda} a_\lambda x^{1-\gamma+\lambda} = x^{1-\gamma}\sum_{\lambda} a_\lambda x^{\lambda} \qquad .... (12)$$

From equation (8) we have

$$a_{\lambda+1} = \frac{(1-\gamma+\lambda+\alpha)(1-\gamma+\lambda+\beta)}{(1-\gamma+\lambda+1)(1-\gamma+\lambda+\gamma)}\, a_\lambda = \frac{(\lambda+\alpha')(\lambda+\beta')}{(\lambda+\gamma')(\lambda+1)}\, a_\lambda \qquad .... (13)$$

Where $\alpha' = 1-\gamma+\alpha,\ \beta' = 1-\gamma+\beta\ and\ \gamma' = 2-\gamma$ .... (14)
Substituting $\lambda = 0, 1, 2, 3.......\ etc$;we get

$$a_1 = \frac{\alpha'.\beta'}{1.\gamma'}\, a_0$$

$$a_2 = \frac{(\alpha'+1)(\beta'+1)}{2(\gamma'+1)}\, a_1 = \frac{\alpha'(\alpha'+1)\beta'(\beta'+1)}{2!\gamma'(\gamma'+1)} a_0$$ and so on.

Substituting these values in (12); we get

$$y = a_0 x^{1-\gamma}[1 + \frac{\alpha'.\beta'}{1.\gamma'}x + \frac{\alpha'(\alpha'+1)\beta'(\beta'+1)}{2!\gamma'(\gamma'+1)} x^2 + ... + \frac{(\alpha')_n.(\beta')_n}{n!.(\gamma')_n} x^n + ...]$$

$$= a_0 x^{1-\gamma} \sum_{n=0}^{\infty} \frac{(\alpha')_n.(\beta')_n}{n!.(\gamma')_n}$$

$$= a_0 x^{1-\gamma} \; {}_2F_1(\alpha', \beta', \gamma' \; ; \; x)$$

$$= a_0 x^{1-\gamma} \; {}_2F_1(1 - \gamma + \alpha, 1 - \gamma + \beta, 2 - \gamma; x) \quad .... (15a)$$

If $a_0 = 1$;the solution becomes

$$y = x^{1-\gamma} \; {}_2F_1(1 - \gamma + \alpha, 1 - \gamma + \beta, 2 - \gamma; x) \quad .... (15b)$$

Thus we got two independent solutions(10) and (15) of hypergeometric differential equation about x=0

When $\gamma = 1$,the two solutions become identical and when γ is negative integer say -n; then

$$a_m = \sum_{m=0}^{\infty} \frac{(\alpha)_m(\beta)_m}{m!\,(-n)_m} a_0$$

While $(-n)_m = (-n)(-n+1)...0, 1..... (-n+m-1) = 0$;thereby giving $a_m = \infty$;thus the solution of the type ${}_2F_1(\alpha, \beta, \gamma\; ; \; x)$cannot be obtained for negative value of γ.thus for $\gamma \neq 1, -1, -2..... (negative\ integers)$;the two solutions for $k = 0$ and $k = 1 - \gamma$are linearly independent and so the general solution of hypergeometric equation can be written as

$$y = A\,{}_2F_1(\alpha, \beta, \gamma\; ; \; x) + Bx^{1-\gamma} \; {}_2F_1(1 - \gamma + \alpha, 1 - \gamma + \beta, 2 - \gamma; x). \quad .... (16)$$ where A and B are arbitrary constants.

Case (b): For the singularity at x=1, the solution is obtained by developing the series about x=1. For this, let us substitute $1 - x = t$; so that $\frac{dt}{dx} = -1$

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = -\frac{dy}{dt}$$

And $$\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dx}(-\frac{dy}{dt}) = \frac{d}{dt}(-\frac{dy}{dt}).\frac{dt}{dx} = \frac{d^2y}{dt^2}$$

Then eq(1) reduces to

$$x(1 - x)\frac{d^2y}{dx^2} + [\gamma - (\alpha + \beta + 1)x]\frac{dy}{dx} - \alpha\beta y = 0$$

$$t(1 - t)\frac{d^2y}{dt^2} + [\gamma - (\alpha + \beta + 1)(-t + 1)](-\frac{dy}{dt}) - \alpha\beta y = 0$$

$$t(1 - t)\frac{d^2y}{dx^2} + [(\alpha + \beta - \gamma + 1) - (\alpha + \beta + 1)t]\frac{dy}{dt} - \alpha\beta y = 0$$

This equation is similar to eq(1) except that $\gamma$is replaced by $\gamma' = \alpha + \beta - \gamma + 1$and x by $t = 1 - x$.hence by similar procedure the roots of indicial equation obtained re 0 and $1 - \gamma' = \gamma - \alpha - \beta$and the corresponding solutions are

$$y = a\,{}_2F_1(\alpha, \beta, \gamma', t)$$

$$y = b\,t^{1-\gamma'}\;{}_2F_1(1 - \gamma' + \alpha, 1 - \gamma' + \beta, 2 - \gamma'; t)$$

The general solution is

$$y = A\,{}_2F_1(\alpha, \beta, \gamma'; t) + B\,t^{1-\gamma'}\;{}_2F_1(1 - \gamma' + \alpha, 1 - \gamma' + \beta, 2 - \gamma'; x)$$

$$= A\,{}_2F_1(\alpha, \beta, \alpha + \beta - \gamma + 1, 1 - x) + B(1 - x)^{\gamma-\alpha-\beta}\;{}_2F_1(\gamma - \alpha, \gamma - \beta, \gamma - \alpha - \beta + 1; 1 - x)$$ ....(17)

Case (c) when $x = \infty$,the series solution will consist of series developed about $x = \infty$.for this,we substitute

$$x = \frac{1}{t}\ i.e\ t = \frac{1}{x}\ or\ \frac{dt}{dx} = -\frac{1}{x^2} = -t^2$$

$$\therefore\quad \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = -t^2\frac{dy}{dt}$$ and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$= \frac{d}{dt}\left(-t^2\frac{dy}{dt}\right)\left(-t^2\right) = t^2\left(2t\frac{dy}{dt} + t^2\frac{d^2y}{dt^2}\right)$$

Making these substitutions in(1); we get

$$x(1 - x)\frac{d^2y}{dx^2} + [\gamma - (\alpha + \beta + 1)x]\frac{dy}{dx} - \alpha\beta\gamma = 0$$

$$t^2(1 - t)\frac{d^2y}{dt^2} + [2t(1 - t) - (\alpha + \beta + 1)t + \gamma t^2]\frac{dy}{dt} + \alpha\beta y = 0 \text{ .... } (18)$$

Let the series of solution of above equation be

$$y = \sum_\lambda a_\lambda t^{k+\lambda} \quad \text{---------}\quad (19)$$

then, $\frac{dy}{dt} = \sum_\lambda a_\lambda (k + \lambda)t^{k+\lambda-1}$

And $\frac{d^2y}{dt^2} = \sum_\lambda a_\lambda (k + \lambda)(k + \lambda - 1)t^{k+\lambda-2}$

Making these substitutions in (18), we get

$$t^2(1 - t)\sum_\lambda a_\lambda (k + \lambda)(k + \lambda - 1)t^{k+\lambda-2}$$

$$+ [2t(1 - t) - (\alpha + \beta + 1)t + \gamma t^2]\sum_\lambda a_\lambda (k + \lambda)t^{k+\lambda-1} + \alpha\beta\sum_\lambda a_\lambda t^{k+\lambda} = 0$$

Or $\sum_{\lambda=0} [(k + \lambda)(k + \lambda - 1) + (2 - \alpha - \beta - 1)(k + \lambda) + \alpha\beta]a_\lambda t^{k+\lambda}$

$$- \sum_{\lambda=0} [(k + \lambda)(k + \lambda - 1) + (2 - \gamma)(k + \lambda)a_\lambda (k + \lambda)t^{k+\lambda-1} = 0$$

This equation is an identity, therefore coefficients of various powers of t must be zero. Equating to zero, the coefficients of lowest power of t i.e. t^k to zero; we get

$$a_0[k(k - 1) + (2 - \alpha - \beta - 1)k + \alpha\beta] = 0$$

Or $\quad a_0(k - \alpha)(k - \beta) = 0$

This in an indicial equation and gives $\quad k = \alpha \ or \ \beta$ (since $a_0 \neq 0$) ....(20)

Again, equating to zero the coefficient of general term $t^{k+\lambda+1}$ (the highest power of t); we get

$$[(k + \lambda)(k + \lambda + 1) + (2 - \alpha - \beta - 1)(k + \lambda + 1) + \alpha\beta]a_{\lambda+1}$$
$$- [(k + \lambda)(k + \lambda - 1) + (2 - \gamma)(k + \lambda)a_\lambda = 0$$

This gives the recurrence relation between the coefficients of $a_\lambda{}' s$

$$a_{\lambda+1} = \frac{(k+\lambda)(k+\lambda-1)+(2-\gamma)(k+\lambda)}{(k+\lambda)(k+\lambda+1)+(1-\alpha-\beta)(k+\lambda+1)+\alpha\beta} a_\lambda$$

$$= \frac{(k+\lambda)(k-\gamma+\lambda+1)}{(k+\lambda+1)+(k+\lambda+1-\alpha-\beta)+\alpha\beta} a_\lambda \quad \text{........ } (21)$$

Choice (1) When $k = \alpha$; equation (21) gives

$$a_{\lambda+1} = \frac{(\alpha+\lambda)(\alpha-\gamma+\lambda+1)}{(\alpha+\lambda+1)+(\alpha+\lambda+1-\alpha-\beta)+\alpha\beta} a_\lambda$$

$$(\alpha + \lambda + 1) + (\alpha + \lambda + 1 - \alpha - \beta) + \alpha\beta = (\alpha + \lambda + 1) + (\lambda + 1 - \beta) + \alpha\beta$$
$$= (\lambda + 1)^2 + \alpha(\lambda + 1) - \beta(\lambda + 1) - \alpha\beta + \alpha\beta$$
$$= (\lambda + 1)^2 + \alpha(\lambda + 1) - \beta(\lambda + 1) = (\lambda + 1)(\lambda + 1 + \alpha - \beta)$$

$$= \frac{(\alpha+\lambda)(\alpha-\gamma+\lambda+1)}{(\lambda+1)+(\alpha-\beta+\lambda+1)} a_\lambda \quad \text{................. } (22)$$

Substituting $\lambda = 0, 1, 2, 3.....$ etc. we get

$\lambda = 0 \qquad a_1 = \frac{\alpha(\alpha-\gamma+1)}{(\alpha-\beta+1)} a_0$

$\lambda = 1$ $\quad a_2 = \frac{(\alpha+1)(\alpha-\gamma+2)}{2(\alpha-\beta+2)} a_1 = \frac{(\alpha+1)(\alpha-\gamma+1)(\alpha-\gamma+2)}{2!\,(\alpha-\beta+1)(\alpha-\beta+2)} a_0$ so on, giving

$$a_n = \frac{(\alpha)_n(\alpha-\gamma+1)_n}{n!\,(\alpha-\beta+1)_n} a_0$$

Substituting these values and $k = \alpha$ in series solution (19); we get

$$y = t^{\alpha} \sum_{\lambda} a_{\lambda} t^{\lambda} = a_0 t^{\alpha} \sum_{\lambda} \frac{(\alpha)_n(\alpha-\gamma+1)_n}{n!\,(\alpha-\beta+1)_n} t^{n}$$

$$= a_0 t^{\alpha}\ {}_2F_1(\alpha, \alpha - \gamma + 1, \alpha - \beta + 1, t)$$

As $t = \frac{1}{x}$;we have the solution $k = \alpha$ as

$$y = a_0 x^{-\alpha}\ {}_2F_1(\alpha, \alpha - \gamma + 1, \alpha - \beta + 1, \tfrac{1}{x}) \qquad \text{.........} (23)$$

choice(ii): when $k = \beta$; we have the series solution

$$y = a_0 x^{-\beta}\ {}_2F_1(\beta, \beta - \gamma + 1, \beta - \alpha + 1, \tfrac{1}{x}) \qquad \text{.........} (24)$$

Hence the general solution for case $x = \infty$ is

$$y = Ax^{-\alpha}\ {}_2F_1(\alpha, \alpha - \gamma + 1, \alpha - \beta + 1, \tfrac{1}{x}) + Bx^{-\beta}\ {}_2F_1(\beta, \beta - \gamma + 1, \beta - \alpha + 1, \tfrac{1}{x}) \qquad \text{.........} (25)$$