Elementary properties of Hypergeometric Functions

(i) Symmetry Property : Hypergeometric function is symmetrical with respect to parameters α, βi.e. it does not change if the parameters α, βare interchanged.

Proof. $${}_2F_1(\alpha, \beta, \gamma\,;\ x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{n!\,(\gamma)_n} x^n$$

$$= \sum_{n=0}^{\infty} \frac{(\beta)_n(\alpha)_n}{n!\,(\gamma)_n} x^n = F(\beta, \alpha, \gamma; x) \qquad \ldots\ldots\ldots\ldots (1)$$

$$\therefore \qquad F\left[{}^{\alpha,\beta}_{\gamma} x\right] = F\left[{}^{\beta,\alpha}_{\gamma} x\right]$$

(ii) Differentiation of Hypergeometric Functions

We have $${}_2F_1(\alpha, \beta, \gamma\,;\ x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{n!\,(\gamma)_n} x^n$$

Differentiating with respect to x; we get

$$\frac{d}{dx}\,{}_2F_1(\alpha, \beta, \gamma\,;\ x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{n!\,(\gamma)_n} n\, x^{n-1}$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(n-1)!\,(\gamma)_n} x^{n-1}$$

$$= \sum_{n=1}^{\infty} \frac{(\alpha)_n(\beta)_n}{(n-1)!\,(\gamma)_n} x^{n-1} \qquad (since\ -1! = \infty)$$

$$= \sum_{m=0}^{\infty} \frac{(\alpha)_{m+1}(\beta)_{m+1}}{m!\,(\gamma)_{m+1}} x^{m} \qquad (where\ m = n-1)$$

But $(\alpha)_{m+1} = \alpha(\alpha+1)(\alpha+2)\ldots\ldots(\alpha+m)$

$= \alpha[(\alpha+1)(\alpha+2)\ldots\ldots(\alpha+1+m-1)] = \alpha(\alpha+1)_m$

Similarly, $(\beta)_{m+1} = \beta(\beta+1)_m \quad and \quad (\gamma)_{m+1} = \gamma(\gamma+1)_m$

$$\therefore \quad \frac{d}{dx}\,{}_2F_1(\alpha, \beta, \gamma\,;\ x) = \sum_{m=0}^{\infty} \frac{\alpha(\alpha+1)_m\,\beta(\beta+1)_m}{m!\,\gamma(\gamma+1)_m} x^m$$

$$= \frac{\alpha.\beta}{\gamma} \sum_{m=0}^{\infty} \frac{(\alpha+1)_m\,(\beta+1)_m}{(\gamma+1)_m} x^m$$

$$= \frac{\alpha.\beta}{\gamma}\ {}_2F_1(\alpha+1, \beta+1, \gamma+1\,;\ x)$$

i.e. $$\frac{d}{dx} F\left[{}^{\alpha,\beta}_{\gamma} x\right] = \frac{\alpha.\beta}{\gamma}\left[{}^{\alpha+1,\beta+1}_{\gamma+1}\ x\right] \qquad \ldots\ldots\ldots\ldots\ldots (2)$$

Similarly

$$\frac{d^2}{dx^2}\ {}_2F_1(\alpha,\beta,\gamma\,;\,x)=\frac{\alpha.\beta}{\gamma}\frac{d}{dx}\ {}_2F_1(\alpha+1,\beta+1,\gamma+1\,;\,x)$$

$$=\frac{\alpha.\beta}{\gamma}.\frac{(\alpha+1)\ \ (\beta+1)}{(\gamma+1)}\ {}_2F_1(\alpha+2,\beta+2,\gamma+2\,;\,x)\quad \text{.........}\ (3)$$

By repeating the process m times; we get

$$\frac{d^m}{dx^m}\ {}_2F_1(\alpha,\beta,\gamma\,;\,x)$$

$$=\frac{\alpha(\alpha+1)....(\alpha+m-1)\ \ \beta(\beta+1)...(\beta+m-1).}{\gamma(\gamma+1).....(\gamma+m-1)}.\ {}_2F_1(\alpha+m,\beta+m,\gamma+m\,;\,x)$$

$$=\frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\ {}_2F_1(\alpha+m,\beta+m,\gamma+m\,;\,x)$$

$$\therefore\quad \frac{d^m}{dx^m}\ F\left[{}^{\alpha,\beta}_{\gamma}x\right]=\frac{(\alpha)_m(\beta)_m}{(\gamma)_m}F\left[{}^{\alpha+m,\beta+m}_{\gamma+m}\ x\right]\quad \text{................}\ (4)$$

Corollary (1) when $x=0$;

$$ {}_2F_1(\alpha,\beta,\gamma\,;\,0)=\lim_{x\to 0}\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{n!\,(\gamma)_n}x^n$$

$$=\lim_{x\to 0}\left[1+\frac{\alpha.\beta}{\gamma}x+\frac{\alpha(\alpha+1)\ \ \beta(\beta+1)}{\gamma\,(\gamma+1)2!}x^2+.....\right]$$

$=1$,since all terms except the first term vanish

Thus $\quad {}_2F_1(\alpha,\beta,\gamma\,;\,0)=1$

Similarly. $\quad {}_2F_1(\alpha+1,\beta+1,\gamma+1\,;\,0)=1$ .... (5)

Hence from (2), it follows that

$$\left[\frac{d}{dx}\ {}_2F_1(\alpha,\beta,\gamma\,;\,x)\right]_{n=0}=\frac{\alpha.\beta}{\gamma}\quad \text{............}\ (6)$$

Corollary (II) If α is a negative parameter, say $\alpha=-p$; then

$$ {}_2F_1(-p,\beta,\gamma\,;\,x)=\sum_{m=0}^{\infty}\frac{(-p)_m(\beta)_m}{m!\,(\gamma)_m}x^m$$

$$=\sum_{m=0}^{p}\frac{(-p)_m(\beta)_m}{m!\,(\gamma)_m}x^m\quad \text{.............}\ (7)\ [since\ the\ terms\ for\ m>p\ vanish]$$

$$(-p)_m=(-p)(-p+1)(-p+2)\ ----\ (-p+m-1)$$
$$(-p)_p=(-p)(-p+1)(-p+2)\ ----\ (-p+p-1)$$

$$(-p)_{p+1} = (-p)(-p+1)(-p+2) ---- (-p+p)$$

Similarly, if β is a negative integer say $\beta = -q$; then

$${}_2F_1(\alpha, -q, \gamma ; x) = \sum_{m=0}^{q} \frac{(\alpha)_m(-q)_m}{m!\,(\gamma)_m} x^m \quad .... (8)$$

Obviously, for negative integers α *or* β the series terminates after a finite number of terms.

In case $\alpha = -p, \gamma = -(p+q)$; *p and q being positive integers; then*

$${}_2F_1(-p, \beta, -p-q ; x) = \sum_{m=0}^{\infty} \frac{(-p)_m(\beta)_m}{m!\,(-p-q)_m} x^m \quad .... (9)$$

Where $(-p)_m = (-p)(-p+1).........(-p+m-1) =$

$$= (-1)^m(p)(p-1).........(p-m+1) = (-1)^m \frac{p!}{(p-m)!}$$

Similarly $(-p-q)_m = (-1)^m \frac{(p+q)!}{(p+q-m)!}$

So that $\frac{(-p)_m}{(-p-q)_m} = \frac{p!}{(p-m)!} . \frac{(p+q-m)!}{(p+q)!}$

$$= \frac{p!(p+q-m)(p+q-m-1)--------(p-m+1)(p-m)!}{(p-m)!(p+q)(p+q-1)------(p+1)p!}$$

$$= [(1 - \frac{m}{p+q})(1 - \frac{m}{p+q-1})...(1 - \frac{m}{p+1})] \quad .... (10)$$

Hence equation (9) becomes

$${}_2F_1(-p, \beta, -p-q ; x) = \sum_{m=0}^{\infty} [(1 - \frac{m}{p+q})(1 - \frac{m}{p+q-1})...(1 - \frac{m}{p+1})] \frac{(\beta)_m}{m!} x^m \quad .... (11)$$

It may be noted that the terms on right hand side do not vanish for $m = 0, 1, 2, 3 ... p$and they vanish for $m = p+1, p+2,... p+q$;but the terms do not vanish for $m = p+q+1, p+q+2.......$;. since the terms corresponding to $m = p+q+1$are

$$(1 - \frac{p+q+1}{p+q}).... (1 - \frac{p+q+1}{p+1}). \frac{(\beta)_{p+q+1}}{(p+q+1)!} x^{p+q+1}$$

which is not zero and all the following terms are non-zero. Therefore for $\alpha = -p, \gamma = -(p+q)$ *or* $\beta = -q, \gamma = -(p+q)$; the series stops at p^{th} and q^{th} terms respectively and again starts at $(p+q+1)^{th}$ term.

(iii) Integral Representation.

We have

$$ {}_2F_1(\alpha, \beta, \gamma\,;\, x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{n!\,(\gamma)_n} x^n \qquad \text{..........} (12) $$

Where $(\alpha)_n = \alpha(\alpha+1)\ldots\ldots(\alpha+n-1) = \frac{\Gamma(\alpha+n)}{\Gamma\alpha}$

$\Gamma\, n = (n-1)(n-2)\,(n-3)\ldots\ldots 3.\,2.\,1 = (n-1)!$
$\Gamma\, \alpha = (\alpha-1)(\alpha-2)\,(\alpha-3)\ldots\ldots 3.\,2.\,1$
$\Gamma\,(\alpha+n) = (\alpha+n-1)(\alpha+n-2)\ldots\ldots\ldots\ldots\ \alpha(\alpha-1)\ldots\ldots 3.\,2.\,1$
$\frac{\Gamma(\alpha+n)}{\Gamma\alpha} = (\alpha+n-1)(\alpha+n-2)\ldots\ldots\ldots\ldots\ \alpha$

And so $(\beta)_n = \frac{\Gamma(\beta+n)}{\Gamma\beta}\ ;\ (\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma\gamma}$

$$ \therefore\ \frac{(\beta)_n}{(\gamma)_n} = \frac{\Gamma(\beta+n)}{\Gamma\beta}\frac{\Gamma\gamma}{\Gamma(\gamma+n)} = \frac{\Gamma\gamma}{\Gamma\beta}\frac{\Gamma(\beta+n)\Gamma(\gamma-\beta)}{\Gamma(\gamma+n)\Gamma(\gamma-\beta)} $$

$$ = \frac{\Gamma\gamma}{\Gamma\beta}\frac{\Gamma(\beta+n)\Gamma(\gamma-\beta)}{\Gamma(\beta+n+\gamma-\beta)\Gamma(\gamma-\beta)} $$

$$ = \frac{\Gamma\gamma}{\Gamma\beta\,\Gamma(\gamma-\beta)}\ B(\beta+n, \gamma-\beta) $$

$$ = \frac{\Gamma\gamma}{\Gamma\beta\Gamma(\gamma-\beta)}\ \int_0^1 (1-t)^{\gamma-\beta-1} t^{\beta+n-1} dt \qquad \text{..........} (13) $$

[since $B\,(m, n) = \beta(m, n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)} = \int_0^1 (1-t)^{n-1} t^{m-1} dt$]

where B represents beta function.

Using (13); equation (12) gives

$$ {}_2F_1(\alpha, \beta, \gamma\,;\, x) = \frac{\Gamma\gamma}{\Gamma\beta\Gamma(\gamma-\beta)} \sum_{n=0}^{\infty} \frac{(\alpha)_n x^n}{n!} \int_0^1 (1-t)^{\gamma-\beta-1} t^{\beta+n-1} dt $$

$$ = \frac{\Gamma\gamma}{\Gamma\beta\,\Gamma(\gamma-\beta)} \int_0^1 (1-t)^{\gamma-\beta-1} t^{\beta-1} \sum_{n=0}^{\infty} \{\frac{(\alpha)_n (xt)^n}{n!}\} dt $$

$$ But \sum_{n=0}^{\infty} \frac{(\alpha)_n (xt)^n}{n!} = 1 + \frac{\alpha(xt)}{1!} + \frac{\alpha(\alpha+1)}{2!}(xt)^2 + \ldots\ldots\ldots\ldots = (1-xt)^{-\alpha} \quad for\ |x| < t $$

$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + --------$

$$\therefore \quad {}_2F_1(\alpha, \beta, \gamma\,;\, x) = \frac{\Gamma\gamma}{\Gamma\beta\Gamma(\gamma-\beta)}\int_0^1 (1-t)^{\gamma-\beta-1} t^{\beta-1}(1-xt)^{-\alpha}\,dt \quad ....\,(14a)$$

$$= \frac{1}{\frac{\Gamma\beta\Gamma(\gamma-\beta)}{\Gamma(\beta+\gamma-\beta)}}\int_0^1 (1-t)^{\gamma-\beta-1} t^{\beta-1}(1-xt)^{-\alpha}\,dt$$

$$= \frac{1}{B\,(\beta,\gamma-\beta)}\int_0^1 (1-t)^{\gamma-\beta-1} t^{\beta-1}(1-xt)^{-\alpha}\,dt \quad ....\,(14b)$$

This represents integral representation of hypergeometric function and is valid for $|x| < 1; \gamma > \beta > 0$.

Corollary (iii) Gauss Formula. If we put x=1 in (14b); we get

$${}_2F_1(\alpha, \beta, \gamma\,;\, 1) = \frac{1}{B\,(\beta,\gamma-\beta)}\int_0^1 (1-t)^{\gamma-\beta-1} t^{\beta-1}(1-t)^{-\alpha}\,dt$$

$$= \frac{1}{B\,(\beta,\gamma-\beta)}\int_0^1 (1-t)^{\gamma-\alpha-\beta-1}\, t^{\beta-1}\,dt$$

$$= \frac{1}{B(\beta,\gamma-\beta)}.\,B(\beta, \gamma-\alpha-\beta) \quad ....\,(15)$$

Replacing Beta Functions by γ-function using the property

$B(m, n) = \frac{\Gamma m\Gamma n}{\Gamma(m+n)}$;we get

$${}_2F_1(\alpha, \beta, \gamma\,;\, 1) = \frac{\Gamma\gamma}{\Gamma\beta\Gamma(\gamma-\beta)}.\,\frac{\Gamma\beta\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\beta+\gamma-\alpha-\beta)}$$

$$= \frac{\Gamma\gamma\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \quad ....\,(16)$$

This expression is called *Gauss Formula.*

Corollary (iv) Vandermond's Theorem: if we put $\alpha = -n$(a negative integer); we get

$${}_2F_1(-n, \beta, \gamma\,;\, 1) = \frac{\Gamma\gamma\,\Gamma(\gamma-\beta+n)}{\Gamma(\gamma+n)\Gamma(\gamma-\beta)} = \frac{(\gamma-1)!(\gamma-\beta+n-1)!}{(\gamma+n-1)!(\gamma-\beta-1)!}$$

$$= \frac{(\gamma-1)!(\gamma-\beta+n-1)........(\gamma-\beta)(\gamma-\beta-1)!}{(\gamma+n-1)(\gamma+n-2)...\gamma(\gamma-1)!(\gamma-\beta-1)!}$$

$$= \frac{(\gamma-\beta)(\gamma-\beta+1)\,...(\gamma-\beta+n-1)}{\gamma.(\gamma+1)..(\gamma+n-2)(\gamma+n-1)}$$

$$= \frac{(\gamma-\beta)_n}{(\gamma)_n}$$

Thus $\quad {}_2F_1(-n, \beta, \gamma;\ 1) = \frac{(\gamma-\beta)_n}{(\gamma)_n}$

This relation is called *Vandermond's theorem*.

Corollary (v) Kummer's Theorem :

From equation 14b

$${}_2F_1(\alpha, \beta, \gamma;\ x) = \frac{1}{\Gamma\,(\beta,\gamma-\beta)}\int_0^1 (1-t)^{\gamma-\beta-1} t^{\beta-1} (1-xt)^{-\alpha}\, dt$$

if we substitute $x = -1$and $\gamma = \beta - \alpha + 1$in(14b); we get

$${}_2F_1(\alpha, \beta, \beta-\alpha+1; -1) = \frac{1}{B(\beta,\beta-\alpha+1-\beta)}\int_0^1 (1-t)^{\beta-\alpha+1-\beta-1}\, t^{\beta-1} (1+t)^{-\alpha}\, dt$$

$$= \frac{1}{B(\beta,1-\alpha)}\int_0^1 (1-t^2)^{-\alpha}\, t^{\beta-1}\, dt$$

$$= \frac{1}{B(\beta,1-\alpha)}\int_0^1 (1-p)^{-\alpha}\, p^{(\beta-1)/2} \frac{dp}{2\sqrt{p}} \qquad (putting\ t^2 = p)$$

$t^2 = p \Rightarrow 2t\, dt = dp$ $dt = \frac{dp}{2\sqrt{p}}$	$\frac{(\beta-1)}{2} - \frac{1}{2} = \frac{\beta-1-1}{2} = \frac{\beta-2}{2}$ $= \frac{\beta}{2} - 1$

$$= \frac{1}{2}\frac{\Gamma(\beta+1-\alpha)}{\Gamma\beta\Gamma(1-\alpha)}.\int_0^1 (1-p)^{-\alpha}\, p^{(\beta/2)-1} dp$$

$$= \frac{1}{2}\frac{\Gamma(\beta-\alpha+1)}{\Gamma\beta\Gamma(1-\alpha)}.B(\frac{\beta}{2},\ 1-\alpha)$$

$$= \frac{1}{2}\frac{\Gamma(\beta-\alpha+1)}{\Gamma\beta\Gamma(1-\alpha)}\frac{\Gamma(\beta/2)\,\Gamma(1-\alpha)}{\Gamma(\frac{\beta}{2}+1-\alpha)}$$

$\frac{1}{2}\frac{\Gamma(\beta-\alpha+1)}{\Gamma\beta\Gamma(1-\alpha)}\frac{\Gamma(\beta/2)\,\Gamma(1-\alpha)}{\Gamma(\frac{\beta}{2}+1-\alpha)} = \frac{1}{2}\frac{\Gamma(\beta-\alpha+1)}{\Gamma\beta}\frac{\Gamma(\beta/2)}{\Gamma(\frac{\beta}{2}+1-\alpha)}$ $= \frac{1}{2}\frac{\Gamma(\beta-\alpha+1)}{(\beta-1)!}\frac{(\beta/2-1)!}{\Gamma(\frac{\beta}{2}+1-\alpha)} = \frac{\Gamma(\beta-\alpha+1)}{\beta(\beta-1)!}\frac{\frac{\beta}{2}(\beta/2-1)!}{\Gamma(\frac{\beta}{2}+1-\alpha)}$

$$= \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta+1)}\frac{\Gamma(\frac{\beta}{2}+1)}{\Gamma(1-\alpha+\frac{\beta}{2})} \qquad \text{............ } (18)$$

This relation is called *Kummer's Theorem*.