**( iv ) Linear Transformations of Hypergeometric Functions :**

The linear relations connecting the hypergeometric function from variable x to some other variable x′ [say (1-x) or x/1-x] are known as linear transformations of the hypergeometric functions.

$$ {}_2F_1(\alpha, \beta, \gamma\,;\, x) = \frac{1}{B\,(\beta, \gamma-\beta)} \int_0^1 (1-t)^{\gamma-\beta-1} t^{\beta-1} (1-xt)^{-\alpha}\, dt $$

If we substitute 1-t=p in(14b); we get

$$ {}_2F_1(\alpha, \beta, \gamma\,;\, x) = \frac{1}{B(\beta, \gamma-\beta)} \int_0^1 (1-p)^{\beta-1} p^{\gamma-\beta-1} \{1 - x(1-p)\}^{-\alpha}\, dp $$

$$ = \frac{(1-x)^{-\alpha}}{B(\beta, \gamma-\beta)} \int_0^1 (1-p)^{\beta-1} p^{\gamma-\beta-1} \{1 - \frac{xp}{x-1}\}^{-\alpha}\, dp $$

$$ = \frac{(1-x)^{-\alpha}}{B(\beta, \gamma-\beta)} . B(\gamma-\beta, \beta)\ {}_2F_1(\alpha, \gamma-\beta, \gamma\,;\, \frac{x}{x-1}) \qquad [using\ 14b] $$

$$ {}_2F_1(\alpha, \beta, \gamma\,;\, x) = \frac{1}{B\,(\beta, \gamma-\beta)} \int_0^1 (1-t)^{\gamma-\beta-1} t^{\beta-1} (1-xt)^{-\alpha}\, dt $$

In the above equation substitute $\frac{x}{x-1}$ in the place $x$

$$ {}_2F_1(\alpha, \gamma-\beta, \gamma\,;\, \frac{x}{x-1}) = \frac{1}{B(\beta, \gamma-\beta)} \int_0^1 (1-p)^{\beta-1} p^{\gamma-\beta-1} \{1 - \frac{xp}{x-1}\}^{-\alpha}\, dp $$

$$ \int_0^1 (1-p)^{\beta-1} p^{\gamma-\beta-1} \{1 - \frac{xp}{x-1}\}^{-\alpha}\, dp = B(\beta, \gamma-\beta)\ {}_2F_1(\alpha, \gamma-\beta, \gamma\,;\, \frac{x}{x-1}) $$

$$ = (1-x)^{-\alpha}\ {}_2F_1(\alpha, \gamma-\beta, \gamma\,;\, \frac{x}{x-1}) \qquad \text{.................} (19) $$

(using symmetry property of Beta function)

Similarly, by symmetry property of hypergeometric function;

We have $\ {}_2F_1(\alpha, \beta, \gamma\,,\, x) = (1-x)^{-\beta}\ {}_2F_1(\gamma-\alpha, \beta, \gamma\,;\, \frac{x}{x-1}) \qquad$ ........... $(20)$

Equations (19) and (20) represents relations of hypergeometric functions of variables $x$ and $\frac{x}{x-1}$

Now we want to find the relations between hypergeometric functions for variables x and 1 -x.

The solution of hypergeometric series about x=0 is

$$ y = A\ {}_2F_1(\alpha, \beta, \gamma\,,\, x) + B\, x^{1-\gamma}\ {}_2F_1(\alpha+1-\gamma, \beta+1-\gamma, 2-\gamma, x) $$

which is convergent for $|x| \le 1$ i.e. in the interval (-1,1).

The solution of hypergeometric equation for x=1 is

$$ y = A\ {}_2F_1(\alpha, \beta, 1+\alpha+\beta-\gamma\,, 1-x) + B\,(1-x)^{\gamma-\alpha-\beta}\ {}_2F_1(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1, 1-x) $$

which is convergent for $|1-x| \le 1$ i.e. in the interval (0,2).

Obviously, the common interval for convergence of both solutions about x=0 and x=1 is (0, 1). In this interval we may get the linear transformation between different hypergeometric functions. Let this linear relationship be represented by

$$ {}_2F_1(\alpha, \beta, \gamma\,,\, x) = A\ {}_2F_1(\alpha, \beta, 1 + \alpha + \beta - \gamma\,, 1 - x) + B(1-x)^{\gamma-\alpha-\beta}\ {}_2F_1(\gamma - \alpha, \gamma - \beta, \gamma - \alpha - \beta + 1, 1 - x) \quad \text{.....}\ (21) $$

If we put x=0, we have

$$ {}_2F_1(\alpha, \beta, \gamma\,;\, 0) = 1 = A\ {}_2F_1(\alpha, \beta, 1 + \alpha + \beta - \gamma\,, 1) + B\ {}_2F_1(\gamma - \alpha, \gamma - \beta, \gamma - \alpha - \beta + 1, 1) $$

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| From corollary (1) when x=0 $\quad {}_2F_1(\alpha, \beta, \gamma\,;\, 0) = 1$ |

From Gauss Formula (16)

$$ {}_2F_1(\alpha, \beta, \gamma\,;\, 1) = \frac{\Gamma\gamma\, \Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} $$

Using Gauss Formula (16), we get

$$ 1 = A\frac{\Gamma(1+\alpha+\beta-\gamma)\ \Gamma(1+\alpha+\beta-\gamma-\alpha-\beta)}{\Gamma(1+\alpha+\beta-\gamma-\alpha)\Gamma(1+\alpha+\beta-\gamma-\beta)} + B\frac{\Gamma(\gamma-\alpha-\beta+1)\ \Gamma(\gamma-\alpha-\beta+1-\gamma+\alpha-\gamma+\beta)}{\Gamma(\gamma-\alpha-\beta+1-\gamma+\alpha)\Gamma(\gamma-\alpha-\beta+1-\gamma+\beta)} $$

$$ \mathrm{I} = A.\frac{\Gamma(1+\alpha+\beta-\gamma)\Gamma(1-\gamma)}{\Gamma(1+\beta-\gamma)\Gamma(1+\alpha-\gamma)} + B.\frac{\Gamma(\gamma-\alpha-\beta+1)\Gamma(1-\gamma)}{\Gamma(1-\beta)\Gamma(1-\alpha)} \qquad \text{.......}\ (22) $$

again substituting x=1 in (21); we get

$$ {}_2F_1(\alpha, \beta, \gamma\,;\, 1) = 1 = A\ {}_2F_1(\alpha, \beta, 1 + \alpha + \beta - \gamma\,, 0) $$

But $\quad {}_2F_1(\alpha, \beta, 1 + \alpha + \beta - \gamma\,, 0) = 1$

$$ \therefore\ A = {}_2F_1(\alpha, \beta, \gamma\,;\, 1) = \frac{\Gamma\gamma\, \Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \qquad \text{...........}\ (23)\ (using\ gauss\ formula) $$

Substituting this value of $A$in (22); we get

$$ \mathrm{I} = \frac{\Gamma\gamma\, \Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}.\frac{\Gamma(1+\alpha+\beta-\gamma)\Gamma(1-\gamma)}{\Gamma(1+\beta-\gamma)\Gamma(1+\alpha-\gamma)} + B.\frac{\Gamma(\gamma-\alpha-\beta+1)\Gamma(1-\gamma)}{\Gamma(1-\beta)\Gamma(1-\alpha)} $$

Using the property of gamma functions

$$ \Gamma p\ \Gamma(1 - p) = \frac{\pi}{sin\, p\pi} : we\ get $$

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| $\Gamma\gamma\Gamma(1 - \gamma) == \frac{\pi}{sin\,\gamma\pi}$ |
| $\Gamma(\gamma - \alpha - \beta)\Gamma(1 + \alpha + \beta - \gamma) = \frac{\pi}{sin\,(\gamma-\alpha-\beta)\pi}$ |
| $\Gamma(\gamma - \beta)\Gamma(1 + \beta - \gamma) = \frac{\pi}{sin\,(\gamma-\beta)\pi}$ |

$$\Gamma(\gamma - \alpha)\Gamma(1 + \alpha - \gamma) = \frac{\pi}{\sin(\gamma-\alpha)\pi}$$

$$I = \frac{\sin\pi(\gamma-\alpha)\sin\pi(\gamma-\beta)}{\sin\pi\gamma\sin\pi(\gamma-\alpha-\beta)} + B\frac{\Gamma(\gamma-\alpha-\beta+1)\Gamma(1-\gamma)}{\Gamma(1-\beta)\Gamma(1-\alpha)}$$

$$\therefore \quad B = \frac{\Gamma(1-\beta)\Gamma(1-\alpha)}{\Gamma(\gamma-\alpha-\beta+1)\Gamma(1-\gamma)}[1 - \frac{\sin\pi(\gamma-\alpha)\sin\pi(\gamma-\beta)}{\sin\pi\gamma\sin\pi(\gamma-\alpha-\beta)}]$$

$\sin\pi\gamma\sin\pi(\gamma - \alpha - \beta) - \sin\pi(\gamma - \alpha)\sin\pi(\gamma - \beta)$

We know $-2\sin A\sin B = \cos(A + B) - Cos(A - B)$

$= -\frac{1}{2}[\cos\pi(\gamma + \gamma - \alpha - \beta) - \cos\pi(\gamma - \gamma + \alpha + \beta) - \cos\pi(\gamma - \alpha + \gamma - \beta)$

$+ \cos\pi(\gamma - \alpha - \gamma + \beta)]$

$= -\frac{1}{2}[-\cos\pi(\alpha + \beta) + \cos\pi(\beta - \alpha)] = \frac{1}{2}[\cos\pi(\alpha + \beta) - \cos\pi(\beta - \alpha)$

We know $\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

$\frac{1}{2}[-2\sin\pi(\frac{\alpha+\beta+\beta-\alpha}{2})\sin\pi(\frac{\alpha+\beta-\beta+\alpha}{2})$

$$= \frac{\Gamma(1-\beta)\Gamma(1-\alpha)\Gamma(\alpha+\beta-\gamma)}{\Gamma(1-\gamma)\Gamma(\alpha+\beta-\gamma)\Gamma(\gamma-\alpha-\beta+1)}[\frac{-\sin\pi\alpha\sin\pi\beta}{\sin\pi\gamma\sin\pi(\gamma-\alpha-\beta)}]$$

$$= \frac{\Gamma(1-\beta)\Gamma(1-\alpha)\Gamma(\alpha+\beta-\gamma)}{\Gamma(1-\gamma)\{\frac{\pi}{\sin\pi(\alpha+\beta-\gamma)}\}}[\frac{-\sin\pi\alpha\sin\pi\beta}{\sin\pi\gamma\sin\pi(\gamma-\alpha-\beta)}]$$

$$= \frac{\Gamma(1-\beta)\Gamma(1-\alpha)\Gamma(\alpha+\beta-\gamma)}{\pi\Gamma(1-\gamma)}\frac{\sin\pi\alpha\sin\pi\beta}{\sin\pi\gamma}$$

$$= \frac{\Gamma(1-\beta)\Gamma(1-\alpha)\Gamma(\alpha+\beta-\gamma)}{\pi\Gamma(1-\gamma)}\frac{\frac{\pi}{\Gamma\alpha\Gamma(1-\alpha)}\cdot\frac{\pi}{\Gamma\beta\Gamma(1-\beta)}}{\frac{\pi}{\Gamma\gamma\Gamma(1-\gamma)}}$$

$$= \frac{\Gamma\gamma\Gamma(\alpha+\beta-\gamma)}{\Gamma\alpha\,\Gamma\beta} \quad \text{..............} (24)$$

Substituting values of $A$ and $B$ in (21); we get

$${}_2F_1(\alpha, \beta, \gamma; x) = \frac{\Gamma\gamma\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}\,{}_2F_1(\alpha, \beta, 1 + \alpha + \beta - \gamma, 1 - x)$$

$$+ \frac{\Gamma\gamma\Gamma(\alpha+\beta-\gamma)}{\Gamma\alpha\,\Gamma\beta}(1 - x)^{\gamma-\alpha-\beta}.\,{}_2F_1(\gamma - \alpha, \gamma - \beta, \gamma - \alpha - \beta + 1, 1 - x) \text{ ...... } (25)$$

This is required relationship