6.48 Confluent Hypergeometric Equation and Function

The confluent hypergeometric equation is

$$x(1-x)\frac{d^2y}{dx^2} + \{\gamma - (\alpha + \beta + 1)x\}\frac{dy}{dx} - \alpha\beta y = 0 \quad \text{........} (1)$$

Replacing x by $\frac{x}{\beta}$; we get

$$\frac{x}{\beta}(1-\frac{x}{\beta})\frac{d^2y}{d(\frac{x}{\beta})^2} + \{\gamma - (\alpha + \beta + 1)\frac{x}{\beta}\}\frac{dy}{d(\frac{x}{\beta})} - \alpha\beta y = 0$$

$$\frac{x}{\beta}(1-\frac{x}{\beta})\beta^2\frac{d^2y}{dx^2} + \{\gamma - (\alpha + \beta + 1)\frac{x}{\beta}\}\beta\frac{dy}{dx} - \alpha\beta y = 0$$

or $$x(1-\frac{x}{\beta})\frac{d^2y}{dx^2} + \{\gamma - (1 + \frac{\alpha+1}{\beta})x\}\frac{dy}{dx} - \alpha y = 0 \quad \text{..............} (2)$$

As ${}_2F_1(\alpha, \beta, \gamma\,; x)$ is the solution of eq(1),the solution of equation (2) is

$${}_2F_1(\alpha, \beta, \gamma\,; \frac{x}{\beta})$$

Now if $\beta \to \infty$,then equation (2) reduces to

$$x\frac{d^2y}{dx^2} + (\gamma - x)\frac{dy}{dx} - \alpha y = 0 \text{.......... } (3)$$

This equation is called *confluent hypergeometric differential equation* and often occurs in boundary value problems of mathematical physics.the solution of eq(3) may be expressed as

$$\lim_{\beta\to\infty} {}_2F_1(\alpha, \beta, \gamma\,; \frac{x}{\beta}) = \lim_{\beta\to\infty} \sum_{m=0}^{\infty} \frac{(\alpha)_m(\beta)_m}{(\gamma)_m m!}(\frac{x}{\beta})^m$$

$$= \lim_{\beta\to\infty} \sum_{m=0}^{\infty} \frac{(\alpha)_m}{(\gamma)_m m!}.\frac{(\beta)_m}{(\beta)^m}x^m$$

But $$\lim_{\beta\to\infty} \frac{(\beta)_m}{(\beta)^m} = \lim_{\beta\to\infty} \frac{\beta(\beta+1)(\beta+2)\text{.......}(\beta+m-1)}{\beta^m}$$

$$= \lim_{\beta\to\infty} (1+\frac{1}{\beta})(1+\frac{2}{\beta})\text{....}(1+\frac{m-1}{\beta}) = 1$$

$$\therefore \lim_{\beta\to\infty} {}_2F_1(\alpha, \beta, \gamma\,; \frac{x}{\beta}) = \sum_{m=0}^{\infty} \frac{(\alpha)_m}{(\gamma)_m}\frac{x^m}{m!} \quad \text{........} (4)$$

This is denoted by ${}_1F_1(\alpha, \gamma\,; x)$ and is known as a confluent *hypergeometric function.* The leading subscript 1 indicates that the first symbol in bracket is numerator and the second subscript 1 indicates that second symbol γ is the denominator.

confluent.hypergeometric equation $x = 0$is a removable (non essential) singularity; so its solution may be developed directly by series method at $x = 0$ taking the series as

$$y = \sum_{r=0}^{\infty} a_r x^{k+r} \qquad \text{..........} (5)$$

So that $$\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (k + r) x^{k+r-1}$$

And $$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (k + r)(k + r - 1) x^{k+r-2}$$

Substituting these values in (3) ; we get

$$x \sum_{r=0}^{\infty} a_r (k + r)(k + r - 1) x^{k+r-2} + (\gamma - x) \sum_{r=0}^{\infty} a_r (k + r) x^{k+r-1} - \alpha \sum_{r=0}^{\infty} a_r x^{k+r} = 0$$

$$\sum_{r=0}^{\infty} a_r (k + r)(k + r - 1) x^{k+r-1} + \gamma \sum_{r=0}^{\infty} a_r (k + r) x^{k+r-1} - \sum_{r=0}^{\infty} a_r (k + r) x^{k+r} - \alpha \sum_{r=0}^{\infty} a_r x^{k+r} = 0$$

$$\sum_{r=0}^{\infty} a_r [(k + r)(k + r - 1 + \gamma) x^{k+r-1} - (k + r + \alpha) x^{k+r} = 0$$

comparing the coefficients of lowest power of x to zero, we get the *indicial equation*

$$a_0 k(k + \gamma - 1) = 0$$

As $a_0 \neq 0$; we get $k = 0 \text{ or } k = 1 - \gamma$

Now comparing the coefficients of the general terms x^{k+m} to zero; we get the *recurrence relation*

$$a_{m+1}(k + m + 1)(k + m + \gamma) - a_m (k + m + \alpha) = 0$$

$$a_{m+1} = \frac{k+m+\alpha}{(k+m+1)(k+m+\gamma)} a^m$$

Which for $k = 0$becomes

$$a_{m+1} = \frac{m+\alpha}{(m+1)(m+\gamma)} a_m \qquad \text{.................} (7)$$

$$for\ m = 0 \Rightarrow a_1 = \frac{\alpha}{\gamma} a_0$$

$$for\ m = 1 \Rightarrow a_2 = \frac{1+\alpha}{(1+1)(1+\gamma)} a_1 = \frac{\alpha}{\gamma} \frac{1+\alpha}{2!(1+\gamma)} a_0$$

$$for\ m = 2 \Rightarrow a_3 = \frac{2+\alpha}{(2+1)(2+\gamma)} a_1 = \frac{2+\alpha}{(2+1)(2+\gamma)} \frac{\alpha}{\gamma} \frac{1+\alpha}{2!(1+\gamma)} a_0 = \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)3!} a_0$$

Hence the solution of *confluent hypergeometric equation* for $k = 0$ becomes

$$y = \sum_{m=0}^{\infty} a_m x^m = a_0[1 + \frac{\alpha}{\gamma}x + \frac{\alpha(\alpha+1)}{2!\,\gamma(\gamma+1)}x^2 + ...]$$

$$= \sum_{m=0}^{\infty} \frac{(\alpha)_m}{(\gamma)_m m!}x^m = a_0\ {}_1F_1(\alpha, \gamma\,; x) \qquad .... (8)$$

And for $k = 1 - \gamma$;the solution is

$$y = a_0 x^{1-\gamma}[1 + \frac{\alpha'}{\gamma'}\,x + \frac{\alpha'\,(\alpha' + 1)}{2!\,\gamma'\,(\gamma' + 1)}x^2 + ...] \qquad [where\ \alpha' = \alpha - \gamma + 1\ and\ \gamma' = 2 - \gamma]$$

$$= a_0 x^{1-\gamma} \sum_{m=0}^{\infty} \frac{(\alpha')_m}{(\gamma')_m m!}x^m = a_0 x^{1-\gamma}\ {}_1F_1(\alpha', \gamma'\ ; x) \qquad .... (9)$$

Where ${}_1F_1(\alpha - \gamma + 1, 2 - \gamma; x)$is called the *confluent hypergeometric function* of second kind.

Therefore the general solution of *confluent hypergeometric equation* is

$$y = A\ {}_1F_1(\alpha, \gamma\,; x) + B\,x^{1-\gamma}\ {}_1F_1(\alpha - \gamma + 1,\ 2 - \gamma, x) \qquad .... (10)$$

This situation holds for $\gamma > 0$.

By the ratio test of $(m + 1)^{th}$term to m^{th}term; we have

$$\left|\frac{u_{m+1}}{u_m}\right| = \left|\frac{(\alpha)_{m+1}}{(\gamma)_{m+1}(m+1)!} \times \frac{(\gamma)_m\,m!}{(\alpha)_m}x\right| = \left|\frac{(\alpha+m)}{(\gamma+m)\,(m+1)}\right| \to 0\ as\ m \to \infty$$

i.e $\left|\frac{u_{m+1}}{u_m}\right| < 1$for all values of$x$.

This shows that the *confluent hypergeometric equation* is convergent.