6.49 Properties of confluent hypergeometric Function

(i) **Differentiation.** We have

$$ {}_1F_1(\alpha, \gamma\,; x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n} \frac{x^n}{n!} \quad \text{........ (1)}$$

Differentiating both sides with respect to x;we get

$$\frac{d}{dx}[\,{}_1F_1(\alpha, \gamma\,; x)] = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n} \frac{n\,x^{n-1}}{n!}$$

$$= \sum_{n=1}^{\infty} \frac{(\alpha)_n}{(\gamma)_n (n-1)!} x^{n-1}$$

$$= \sum_{m=0}^{\infty} \frac{(\alpha)_{m+1}}{(\gamma)_{m+1}\, m!}\, x^m \qquad (putting\; n-1=m)$$

$$= \sum_{m=0}^{\infty} \frac{\alpha(\alpha+1)_m}{\gamma(\gamma+1)_m} \frac{x^m}{m!} \qquad [since\; \alpha_{m+1} = \alpha(\alpha+1)_m]$$

$$\therefore \qquad \frac{d}{dx}[\,{}_1F_1(\alpha, \gamma\,; x)] = \frac{\alpha}{\gamma}\; {}_1F_1(\alpha+1, \gamma+1\,; x) \qquad \text{........ (2)}$$

Differentiating again with respect to x, we get

$$\frac{d^2}{dx^2}[\,{}_1F_1(\alpha, \gamma\,; x)] = \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\; {}_1F_1(\alpha+2, \gamma+2; x) \quad \text{........ (3)}$$

Proceeding similarly; we get

$$\frac{d^p}{dx^p}[\,{}_1F_1(\alpha, \gamma\,; x)] = \frac{(\alpha)_p}{(\gamma)_p}\; {}_1F_1(\alpha+p, \gamma+p; x) \quad \text{........... (4)}$$

(ii) if αis negative integer, the series terminates after a finite number of terms

(iii) If γ is also a negative integer, the series which terminates at a certain term for α,restarts again after certain terms.

The proof of (ii) and (iii) is analogous to that given for hypergeometric function.

(iv) **Integral Representation:** we have

$$ {}_1F_1(\alpha, \gamma\,; x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n} \frac{x^n}{n!}$$

Where $(\alpha)_n = \alpha(\alpha+1)\ldots\ldots(\alpha+n-1) = \frac{\Gamma(\alpha+n)}{\Gamma\alpha}$

$$\therefore \qquad \frac{(\alpha)_n}{(\gamma)_n} = \frac{\Gamma(\alpha+n)}{\Gamma\alpha}.\frac{\Gamma\gamma}{\Gamma(\gamma+n)} = \frac{\Gamma\gamma}{\Gamma\alpha\Gamma(\gamma-\alpha)}.\frac{\Gamma(\gamma-\alpha)\,\Gamma(\alpha+n)}{\Gamma(\alpha+n+\gamma-\alpha)}$$

$$= \frac{\Gamma\gamma}{\Gamma\alpha\Gamma(\gamma-\alpha)} \beta(\alpha + n, \gamma - \alpha)$$

$$= \frac{\Gamma\gamma}{\Gamma\alpha\Gamma(\gamma-\alpha)} \int_0^1 t^{\alpha+n-1} (1 - t)^{\gamma-\alpha-1}\, dt \quad \text{.......... (5)}$$

[since Beta Function $\beta(m,\ n) = \frac{\Gamma m\, \Gamma n}{\Gamma(m+n)} = \int_0^1 t^{m-1} (1 - t)^{n-1}\, dt$ thus from (1); we have

$${}_1F_1(\alpha, \gamma\,; x) = \sum_{n=0}^{\infty} \frac{\Gamma\gamma}{\Gamma\alpha\Gamma(\gamma-\alpha)} \int_0^1 t^{\alpha+n-1} (1 - t)^{\gamma-\alpha-1} \frac{x^n}{n!}\, dt$$

$$= \frac{\Gamma\gamma}{\Gamma\alpha\Gamma(\gamma-\alpha)} \int_0^1 t^{\alpha-1} (1 - t)^{\gamma-\alpha-1} \left(\sum_{n=0}^{\infty} \frac{(xt)^n}{n!}\right) dt$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma\gamma}{\Gamma\alpha\Gamma(\gamma-\alpha)} \int_0^1 t^{\alpha-1} (1 - t)^{\gamma-\alpha-1} e^{xt}\, dt \quad \text{................ (6)}$$

$$\therefore \qquad {}_1F_1(\alpha, \gamma\,; x) = \frac{1}{\beta(\alpha,\gamma-\alpha)} \int_0^1 e^{xt}\, t^{\alpha-1} (1 - t)^{\gamma-\alpha-1}\, dt \quad \text{.................... (7)}$$

This is the integral formula representing the *confluent hypergeometric function.*

Cor. (I) if we substitute $t = 1 - p$in eq(6); we get

$${}_1F_1(\alpha, \gamma\,; x) = \frac{\Gamma\gamma}{\Gamma\alpha\Gamma(\gamma-\alpha)} \int_0^1 e^{x(1-p)} (1 - p)^{\alpha-1} p^{\gamma-\alpha-1}\, dp$$

$$= \frac{\Gamma\gamma}{\Gamma\alpha\Gamma(\gamma-\alpha)} e^{x} \int_0^1 e^{-xp} p^{\gamma-\alpha-1} (1 - p)^{\alpha-1} dp$$

$$= \frac{\Gamma\gamma}{\Gamma\alpha\Gamma(\gamma-\alpha)} e^{x}\, \frac{\Gamma\alpha\Gamma(\gamma-\alpha)}{\Gamma\gamma}\ {}_1F_1(\gamma - \alpha, \gamma\,; - x) \quad [using\ (6)]$$

$$\therefore \qquad {}_1F_1(\alpha, \gamma\,; x) = e^{x}\ {}_1F_1(\gamma - \alpha, \gamma\,; - x)$$

This relation is called *Kummer's* relation.

Cor (II) if we substitute $x = 0$in eq(6), we get

$${}_1F_1(\alpha, \gamma\,; x) = \frac{\Gamma\gamma}{\Gamma\alpha\Gamma(\gamma-\alpha)} \int_0^1 t^{\alpha-1} (1 - t)^{\gamma-\alpha-1} dt$$

$$= \frac{\Gamma\gamma}{\Gamma\alpha\Gamma(\gamma-\alpha)} \beta(\alpha, \gamma - \alpha)$$

$$= \frac{\Gamma\gamma}{\Gamma\alpha\Gamma(\gamma-\alpha)} . \frac{\Gamma\alpha\Gamma(\gamma-\alpha)}{\Gamma\gamma} = 1$$

$$\therefore \qquad {}_1F_1(\alpha, \gamma\,; x) = 1 \quad \text{........... (9)}$$