6:50. Representation of Various Functions in terms of confluent Hypergeometric Functions.

Various familiar functions of mathematical physics may be expressed as the particular cases of the *confluent hypergeometric functions* corresponding to suitable choices of parameters α, γ and variable x.

(1) Elementary Functions. We have

$$ {}_1F_1(\alpha, \gamma ; x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n n!} x^n \quad \text{........ (1)} $$

Substituting $\gamma = \alpha$, we get

$$ {}_1F_1(\alpha, \alpha; x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + ... = e^x $$

Thus $$ e^x = {}_1F_1(\alpha, \alpha; x) \quad \text{.......... (2)} $$

$$ {}_1F_1(\alpha + 1, \alpha; x) = \sum_{n=0}^{\infty} \frac{(\alpha+1)_n}{(\alpha)_n n!} x^n = 1 + x + \frac{x^2}{2!} + ... = e^x $$

$$ (\alpha + 1)_n = (\alpha + 1)\ (\alpha + 2)\ ----- \ (\alpha + 1 + n - 1) $$

$$ (\alpha + 1)_n = \frac{1}{\alpha} \alpha(\alpha + 1)\ (\alpha + 2)\ ----- \ (\alpha + n - 1)(\alpha + n) $$

$$ = \frac{1}{\alpha} (\alpha)_n (\alpha + n) $$

$$ {}_1F_1(\alpha + 1, \alpha; x) = \sum_{n=0}^{\infty} \frac{(\alpha+n)(\alpha)_n}{\alpha(\alpha)_n n!} x^n = \sum_{n=0}^{\infty} \frac{(\alpha+n)}{\alpha} \frac{x^n}{n!} = (1 + \frac{n}{\alpha}) \frac{x^n}{n!} [1 + x + \frac{x^2}{2!} + ...] $$

$$ =[1 + x + \frac{x^2}{2!} + ...] + \frac{1}{\alpha} (x + \frac{2x^2}{2!} + \frac{3x^3}{3!} .....] = e^x + \frac{x}{\alpha} (1 + x + \frac{x^2}{2!} .....] $$

Similarly $$ {}_1F_1(\alpha + 1, \alpha; x) = (1 + \frac{x}{\alpha}) e^x \quad \text{.......... (3)} $$

If we substitute $\alpha = 1,\ \gamma = 2$in eq (1); we get

$$ {}_1F_1(1, 2; x) = \sum_{n=0}^{\infty} \frac{(1)_n}{(2)_n n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!} $$

$$ (1)_n = 1.2.3\ ----- \ (1 + n - 1) = n! $$

$$ (2)_n = 2.3\ ----- \ (2 + n - 1) = (n + 1)! $$

$$= \frac{1}{x}\left[x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right]$$

$$1 + \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + ..\right) - 1 = e^x - 1$$

$$= \frac{e^x - 1}{x} \qquad \ldots\ldots\ldots (4)$$

If $\alpha = -2, \gamma = 1$; then

$${}_1F_1(-2, 1; x) = \sum_{n=0}^{\infty} \frac{(-2)_n}{(1)_n\, n!} x^n = 1 - 2x + \frac{x^2}{2} \qquad \ldots\ldots\ldots (5)$$

$$\sum_{n=0}^{\infty} \frac{(-2)_n}{(1)_n\, n!} x^n = 1 + \frac{(-2)}{(1)\ 1!} x + \frac{(-2)(-1)}{(1)(2)\ \ 2!} x^2 + \frac{(-2)(-1)(0)}{(1)(2)(3)\ \ 3!} x^3 + \ldots\ldots$$

Equations (2), (3), (4), (5) represent various elementary functions in terms of *Hypergeometric functions.*

(2) Hermite Polynomials: we have

$$H_n(x) = \sum_{r=0}^{p} (-1)^r \frac{n!}{r!\,(n-2r)!} (2x)^{n-2r} \qquad \ldots\ldots\ldots\ldots (9)$$

[where $p = \frac{n}{2}$ if n is even, $p = \frac{n-1}{2}$ if n is odd]

From this even order *Hermite Polynomials* can be written as

$$H_{2n}(x) = \sum_{r=0}^{n} (-1)^r \frac{(2n)!}{r!\,(2n-2r)!} (2x)^{2n-2r}$$

$$= (2n)!\ (-1)^n \sum_{r=0}^{n} \frac{(-1)^r (2x)^{2r}}{(n-r)!\,(2r)!}$$

$$\sum_{r=0}^{n} \frac{(-1)^r}{r!\,(2n-2r)!} (2x)^{2n-2r} = \frac{(-1)^0}{0!\,(2n-2.0)!} (2x)^{2n-2.0} + \frac{(-1)}{1!\,(2n-2)!} (2x)^{2n-2}$$

$$+ \frac{(-1)^2}{2!\,(2n-4)!} (2x)^{2n-4} + \ldots\ldots + \frac{(-1)^{n-1}}{(n-1)!\,[2n-2(n-1)]!} (2x)^{2n-2(n-1)}$$

$$+ \frac{(-1)^n}{n!\,[2n-2n]!}(2x)^{2n-2n}$$

Now right to left

$$= \frac{(-1)^n}{n!\,[2n-2n]!}(2x)^{2n-2n} + \frac{(-1)^{n-1}}{(n-1)!\,[2n-2(n-1)]!}(2x)^{2n-2(n-1)}$$

$$+ \frac{(-1)^{n-2}}{(n-2)!\,[2n-2(n-2)]!}(2x)^{2n-2(n-2)} + ---$$

$$= \frac{(-1)^n}{n!\,[0]!}(2x)^0 + \frac{(-1)^n(-1)^1}{(n-1)!\,[2.1)]!}(2x)^{2.1} + \frac{(-1)^n(-1)^2}{(n-2)!\,[2.2]!}(2x)^{2.2} + ---$$

$$= (-1)^n \sum_{r=0}^{n} \frac{(-1)^r (2x)^{2r}}{(n-r)!\,(2r)!}$$

$$(-n)_r = -n(-n+1)(-n+2)(-n+3)\ ---\ (-n+r-1)$$

$$= (-1)^r\, n(n-1)(n-2)(n-3) + ---- + (n-r+1)$$

$$= (-1)^r\, n(n-1)(n-2)(n-3) + ---- + (n-r+1)\frac{(n-r)!}{(n-r)!}$$

$$= (-1)^r \frac{n!}{(n-r)!}$$

$$\therefore \frac{(-1)^r}{(n-r)!} = \frac{(-n)_r}{n!}$$

$$= (-1)^n \frac{(2n)!}{n!} \sum_{r=0}^{n} \frac{(-n)_r}{(2r)!}(2x)^{2r}$$

$$= (-1)^n \frac{(2n)!}{n!} \sum_{r=0}^{n} \frac{(-n)_r (x^2)^r}{(\frac{1}{2})_r r!} \qquad [since\ (2r)! = 2^{2r}(\tfrac{1}{2})_r r!]$$

$$(\tfrac{1}{2})_r = (\tfrac{1}{2})\ (\tfrac{1}{2}+1)\ (\tfrac{1}{2}+2)\ + ----\ (\tfrac{1}{2}+r-1)$$

$$= (\tfrac{1}{2})\ (\tfrac{3}{2})\ (\tfrac{5}{2})\ + -----\ (\tfrac{1}{2}+r-2)\ (\tfrac{1}{2}+r-1)$$

$$= (\tfrac{1}{2})\ (\tfrac{3}{2})\ (\tfrac{5}{2})\ + -----\ (\tfrac{2r-3}{2})\ (\tfrac{2r-1}{2})$$

$$(2r)! = (2r)(2r-1)(2r-2)(2r-3)(2r-4)\ -----\ 2.1)$$

$$= [(2r)(2r-2)(2r-4)\ ---\ 6.4.2)][(2r-1)(2r-3)\ -----\ 5.3.1]$$

$$= 2^r[(r)(r-1)(r-2)\ ---\ 3.2.1)]2^r[(\tfrac{2r-1}{2})(\tfrac{2r-3}{2})\ -----\ (\tfrac{5}{2})(\tfrac{3}{2})\ (\tfrac{1}{2})\ ]$$

$$= 2^{2r}(\tfrac{1}{2})_r r!$$

$$= (-\ 1)^n \frac{(2n)!}{n!}\ {}_1F_1(-\ n, \tfrac{1}{2}, x^2) \quad .... (10)$$

Again *odd order Hermite Polynomials* may be expressed as

$$H_{2n+1}(x) = \sum_{r=0}^{(2n+1)/2} \frac{(-1)^r (2n+1)!}{r!\,(2n+1-2r)!} (2x)^{2n+1-2r}$$

$$= (-\ 1)^n \frac{(2n+1)!}{n!}.\ 2x \sum_{r=0}^{n} \frac{(-n)_r . x^{2r}}{(\frac{3}{2})_r\ r!}$$

$$= (-\ 1)^n \frac{(2n+1)!}{n!}.\ 2x.\ \ {}_1F_1(-\ n, \tfrac{3}{2}, x^2) \quad ...... (11)$$

(3) Laguerre Polynomials: the Laguerre Polynomials are defined

$$L_n(x) = \sum_{r=0}^{n} (-\ 1)^r \frac{(n!)}{(n-r)!\,(r!)^2} x^r \quad .... (12)$$

But $\frac{(-1)^r}{(n-r)!} = \frac{(-1)^r n(n-1)(n-2).....(n-r+1)}{n!}$

$$= \frac{(-n)(-n+1)(-n+2).....(-n+r+1)}{n!}$$

$$= \frac{(-n)_r}{n!}$$

$$\therefore \quad L_n(x) = \sum_{r=0}^{n} \frac{(-n)_r}{n!} \frac{(n)!}{(r!)^2} x^r$$

$$= \sum_{r=0}^{n} \frac{(-n)_r}{(r!)^2}\ x^r$$

$$= \sum_{r=0}^{n} \frac{(-n)_r}{(1)_r\ r!}\ x^r$$

$$(1)_r = 1.2.3 \ ----- \ (1 + r - 1) = r!$$

$$= \ {}_1F_1(-\ n, 1; x) \quad .... (13)$$