Fourier's and Laplace's Integral Transforms

In mathematical physics we used pairs of functions like g(α) = $\int_a^b f(t)\, k(\alpha, t)\, dt$

The function g(α) is called the integral transform of f(t), by the kernel $k(\alpha, t)$.

The integral transforms are useful in mathematical analysis and physical applications.

g(α) = $\int_0^\infty f(t)\, e^{-i\omega t}\, dt$ ……. Fourier Transform

g(α) = $\int_0^\infty f(t)\, e^{-\alpha t}\, dt$ ……… Laplace Transform

Fourier's Transform

If $f(x)$is periodic function of x, the the *Fourier Integral* of $f(x)$may be expressed

$$f(x) = \frac{1}{2\pi} \int_{+\infty}^{-\infty} e^{i\omega x} d\omega \int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t}$$

This may be expressed as

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{+\infty}^{-\infty} e^{i\omega x} d\omega \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\omega x} g(\omega)\, d\omega \quad \text{.........} (1)$$

where $$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t} \quad \text{.........} (2)$$

The function $g(\omega)$is called the *Fourier Transform* of $f(t)$and $f(t)$is called *Fourier Inverse Transform* of $g(\omega)$.

The integral (2) transforms a time function $f(t)$into its equivalent frequency function $g(\omega)$;while integral (1) reverses the process.

Infinite Fourier Sine and Cosine Transforms. The Fourier Transformation of $f(t)$ is given by

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} f(t)\, e^{-i\omega t}\, dt + \int_0^\infty f(t)\, e^{-i\omega t}\, dt \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_0^\infty f(-t)\, e^{i\omega t}\, dt + \int_0^\infty f(t)\, e^{-i\omega t}\, dt \right] \quad \text{........} (3)$$

(Replacing t by -t in first integral)

Now $f(t) = \{ {f(-t) \text{ if function } f(t) \text{ is even} \atop -f(-t) \text{ if function } f(t) \text{ is odd}}$ .... (4)

Thus eq(3) gives

$$g(\omega) = \{ {\frac{1}{\sqrt{2\pi}} \int_0^\infty f(t)\, (e^{i\omega t} + e^{-i\omega t})\, dt \text{ for even function } f(t) \atop \frac{1}{\sqrt{2\pi}} \int_0^\infty f(t)\, (e^{-i\omega t} - e^{i\omega t})\, dt \text{ for odd function } f(t)}$$

Now using $\frac{e^{i\omega t} + e^{-i\omega t}}{2} = \cos \omega t$ and $\frac{e^{i\omega t} - e^{-i\omega t}}{2i} = \sin \omega t$; we get

$$g(\omega) = \{ {\frac{1}{\sqrt{2\pi}} .2 \int_0^\infty f(t) \cos \omega t\, dt \text{ for even function } ....(5) \atop \frac{1}{\sqrt{2\pi}} . \frac{2}{i} \int_0^\infty f(t) \sin \omega t\, dt \text{ for odd function } ....(6)}$$

The integral (5) is called *Infinite Fourier cosine transform* and (6) is called *infinite sine integral* and are denoted by

$$g_c(\omega) = \sqrt{(\frac{2}{\pi})} \int_0^\infty f(t) \cos \omega t\, dt \quad .... (7)$$

And $$g_s(\omega) = \sqrt{(\frac{2}{\pi})} \int_0^\infty f(t) \sin \omega t\, dt \quad .... (8)$$

The *Inverse Fourier cosine transforms* leat to functions

$$f(x) = \sqrt{(\frac{2}{\pi})} \int_0^\infty g_c(\omega) \cos \omega x\, d\omega \quad .... \quad (9)$$

$$f(x) = \sqrt{(\frac{2}{\pi})} \int_0^\infty g_s(\omega) \sin \omega x\, d\omega \quad .... \quad (10)$$

Equations (7) and (9) form a pair of *Fourier cosine transforms* while equations (8) and (10) form a pair of *Fourier sine transforms*.

Properties of Fourier’s Transform

1. **Addition Theorem or Linearity Theorem.** If $f(t) = a_1 f_1(t) + a_2 f_2(t) +$............. then the Fourier Transforms of $f(t)$is

$$g(\omega) = a_1 g_1(\omega) + a_2 g_2(\omega) + ...$$

Where $g_1(\omega), g_2(\omega)$..are *Fourier Transforms* of $f_1(t),\ f_2(t)$............and a_1, a_2....….. are constants.

Proof. The fourier transform of f(t) is given by

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} [a_1 f_1(t) + a_2 f_2(t) + ...]\, e^{-i\omega t} dt$$

$$= a_1 . \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f_1(t)\, e^{-i\omega t} dt + a_2 . \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f_2(t)\, e^{-i\omega t} dt + ...$$

$$= a_1 g_1(\omega) + a_2 g_2(\omega) + ... \qquad ..... (1)$$

2. **Similarity Theorem or Change of Scale Property:** if $g(w)$is the *Fourier transform* of $f(t)$,*the Fourier Transform* of $f(at)$is

$$\frac{1}{a}\, g(\frac{\omega}{a})$$

Proof. Denoting the *Fourier Transform* of $f(t)$by $F.T\,\{f(t)\}$,we have

$$F.T[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t} dt = g(\omega)$$

Hence $$F.T[f(at)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(at)\, e^{-i\omega t} dt$$

Substituting $y = at$,in above integral, we get

$$F.T[f(at)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y)\, e^{-i\omega y/a} \frac{dy}{a}$$

$$= \frac{1}{a} . \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y)\, e^{-i\omega y/a} dy$$

$$= \frac{1}{a}\, g(\frac{\omega}{a}) \qquad .... (2)$$

This theorem is well known in its applications to waveforms and spectra, where compression of time scale by given factor compresses the periods of all harmonic components equally and therefore increases the frequency of every component by the same factor.