3. *if $g(\omega)$ is the Fourier Transform of $f(t)$, then the Fourier transform of the complex conjugate of $f(t)$ will be given by $g*(-\omega)$; where $*$ indicates the complex conjugate of the corresponding complex function.*

Proof. We have $g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t} dt$

Taking complex conjugate on both sides; we get

$$g*(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f*(t)\, e^{+i\omega t} dt$$

$$g*(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f*(t)\, e^{-i(-\omega)t} dt$$

Replacing ω by $(-\omega)$; we get

$$g^{*}(-\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f*(t)\, e^{-i\omega t} dt$$

Hence $g*(-\omega) = F.T[f*(t)]$ .... (3)

4. Shifting Property:

if $g(\omega)$ is the Fourier Transform of $f(t)$, then the Fourier transform of $f(t \pm a)$ will be given by $e^{\pm i\omega a} g(\omega)$; where a is any constant.

Proof. By definition if Infinite Fourier transform

$$F.T\,[f(t \pm a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t \pm a)\, e^{-i\omega t} dt$$

Substituting $(t \pm a) = y$ i.e, $dt = dy$; we have

$$F.T\,[f(t \pm a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y)\, e^{-i\omega(y \pm a)} dy$$

$$= e^{\pm i\omega a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y)\, e^{-i\omega y} dy$$

$$= e^{\pm i\omega a} g(\omega) \quad .... (4)$$

According to this theorem if a given function be shifted in the positive or negative direction by an amount a, no Fourier component changes in amplitude ; by its Fourier transform suffers phase changes.

5. Modulation Theorem:

if $g(\omega)$ is the Fourier Transform of $f(t)$, then the Fourier transform of $f(t) \cos at$ is given by

$$\frac{1}{2} g(\omega - a) + \frac{1}{2} g(\omega + a)$$

Proof. $F.T\,[f(t) \cos at] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) \cos at.\, e^{-i\omega t} dt$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) \left(\frac{e^{iat} + e^{-iat}}{2}\right) e^{-i\omega t} dt$$

$$= \frac{1}{2} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i(\omega - a)t} f(t) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i(\omega + a)t} f(t)\, dt\right]$$

$$= \frac{1}{2} g(\omega - a) + \frac{1}{2} g(\omega + a) \quad .... (5)$$

6. **Convolution Theorem:** the transform of a product of two functions is given by a convolution integral.
Proof: Let $f_1(t)\ and\ f_2(t)$ be two given functions and their product functions $f(t)$i.e,
$f(t) = f_1(t).f_2(t)$

From definition $$F.T[f(t)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f_1(t).f_2(t)\,e^{-i\omega t}dt \quad .... (6)$$

If $g_1(\omega')$is the Fourier Transform of $f_1(t)$, then the Fourier inverse Transform $g_1(\omega')$ is

$$f_1(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-i\omega' t}g_1(\omega')\,d\omega' \quad .... (7)$$

Substituting value of $f_1(t)$ from (7) in (6); we get

$$F.T[f\ (t)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g_1(\omega')\ e^{-i\omega' t}\ d\omega'\}\,f_2(t).e^{-i\omega t}dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\{g_1(\omega')\int_{-\infty}^{+\infty} f_2(t).e^{-(\omega-\omega')t}dt\}d\omega' \quad .... (8)$$

Now, the Fourier Transform of $f_2(t)$is given by

$$g_2(\omega) = \int_{-\infty}^{+\infty} f_2(t).e^{-i\omega t}dt$$

Replacing $\omega$by $\omega - \omega'$ in above equation; we get

$$g_2(\omega - \omega') = \int_{-\infty}^{+\infty} f_2(t).e^{-i(\omega-\omega')t}dt \quad .... (9)$$

Combining (8) and (9), the Fourier transform of $f(t)$becomes

' $$F.T[f(t)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g_1(\omega').g_2(\omega - \omega')\,d\omega' \quad .... (10)$$

Thus the Fourier Transform of a product of two functions $f_1(t)$ and $f_2(t)$is given by an integral, known as *Convolution Integral* where the functions $g_1$and g_2 are said to be convolved with each other.

7. **Parseval's Theorem:** The Fourier Transform of a Convolution integral is given by the product of transforms of the convolving functions.
Proof: let $f(t)$be given convolution integral i.e,

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f_1(t').f_2(t - t')\,dt' \quad .... (11)$$

The Fourier transform of $f(t)$is

$$g(\omega) = F.T[f(t)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(t)\,e^{-i\omega t}dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f_1(t').f_2(t - t')\,dt'\}.e^{-i\omega t}dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f_1(t')\,e^{-i\omega t'}dt'.\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}.f_2(t - t').e^{-i\omega t}e^{i\omega t'}dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f_1(t') \, e^{-i\omega t'} \, dt' \, . \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f_2(t - t') \, . \, e^{i\omega(t-t')} \, dt \; .... (12)$$

If $g_1(\omega)$and$g_2(\omega)$are Fourier Transforms of $f_1(t)$ and $f_2(t)$respectively, we have

$$g_1(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f_1(t') \, e^{-i\omega t'} \, dt' \qquad .... (13)$$

$$g_2(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f_2(t) \, . \, e^{-i\omega t} dt \qquad .... (14)$$

Changing t and $t - t'$ in (14); we get

$$g_2(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f_2(t - t') \, . \, e^{-i\omega(t-t')} dt \quad .... (15)$$

Hence, from (12),(13) and (15), we have

$$g(\omega) = g_1(\omega). \, g_2(\omega) \qquad .... (16)$$