The Beta Function

The first Eulerian function is generally known as Beta function $\beta(m, n)$

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \qquad \begin{cases} m>0 \\ n>0 \end{cases} \qquad \text{..........} \ (1)$$

The Gamma Function

The second Eulerian function is generally known as Gamma function Γn

$$\Gamma n = \int_0^\infty e^{-x}x^{n-1}dx \qquad n>o$$

Symmetry Property of Beta Function

$$\beta(m, n) = \beta(n, m)$$

By definition $\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \begin{cases} m>0 \\ n>0 \end{cases}$ .... (1)

Substituting x=1 -y $\therefore$ dx =- dy in above equation, we get

$x = 0 \Rightarrow y = 1 \; and \; x = 1 \Rightarrow y = 0$

$$\beta(m, n) = \int_1^0 y^{n-1}(1-y)^{m-1}(-dy) = \int_0^1 y^{n-1}(1-y)^{m-1}dy$$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}dx = \beta(m, n) \qquad \text{..........} \ (2)$$

i.e., Beta Function $\beta(m, n)$ is symmetric with respect to m and n. This property is called the symmetry property of Beta function

Evaluation of Beta Function

By definition $\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \begin{cases} m>0 \\ n>0 \end{cases}$

Integrating the parts by keeping $(1-x)^{n-1}$ as first function, we have

$$\beta(m, n) = \left[(1-x)^{n-1}\frac{x^m}{m}\right]_0^1 + \int_0^1 (n-1)(1-x)^{n-2}\frac{x^m}{m}dx$$

$$= \frac{n-1}{m}\int_0^1 (1-x)^{n-2}x^m dx$$

Integrating again by parts, we get

$$\beta(m, n) = \frac{(n-1)}{m}\left[\left[(1-x)^{n-2}\frac{x^{m+1}}{m+1}\right]_0^1 + \int_0^1 \frac{x^{m+1}}{m+1}(n-2)(1-x)^{n-3}dx\right]$$

$$\beta(m, n) = \frac{(n-1)(n-2)}{m(m+1)} \int_0^1 x^{m+1}(1-x)^{n-3}dx$$

Continuing the process of integration by parts and assuming that n is a positive integer, we obtain

$$\beta(m, n) = \frac{(n-1)(n-2)........ 2.1}{m(m+1)......(m+n-2)} \int_0^1 x^{m+n-2}dx$$

$$= \frac{(n-1)(n-2)........ 2.1}{m(m+1)......(m+n-2)} \left[\frac{x^{m+n-1}}{m+n-1}\right]_0^1$$

$$= \frac{(n-1)!}{m(m+1)......(m+n-2)(m+n-1)} \quad .... (1)$$

Again, if m is also a positive integer, then

$$\beta(m, n) = \frac{(n-1)!(m-1)!}{(m+n-1)!} \quad .... (2)$$

In case m alone is a positive integer, then in view of the symmetry property $\beta(m, n) = \beta(n, m)$, we have

$$\beta(m, n) = \frac{(m-1)!}{n(n+1)........(n+m-1)} \quad .... (3)$$

Transformation of Beta Function (other forms of beta function)

By Definition $\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$ .... (1)

(a) Substituting $x = \frac{y}{1+y}$ $\therefore$ $dx = \frac{dy}{(1+y)^2}$

and $1 - x = \frac{1}{1+y}$, we get

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

$$\frac{d}{dx}\left(\frac{y}{1+y}\right) = \frac{(1+y)dy - ydy}{(1+y)^2} = \frac{dy}{(1+y)^2}$$

$$\beta(m, n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m-1}} \cdot \frac{1}{(1+y)^{n-1}} \cdot \frac{dy}{(1+y)^2}$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}dy \quad .... (2)$$

(b) Also since $\beta(m, n) = \beta(n, m)$

$$\therefore \quad \beta(m, n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}dy \quad .... (3)$$

This equation can be obtained directly from equation (1) by substituting $x = \frac{1}{1+y}$.

Equations (2) and (3). Represent transformed (other forms of Beta Function).