Evaluation of Gamma Function

From definition $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$

Integrating by parts keeping x^{n-1} as first function, we get

$$\Gamma n = [- e^{-x}.x^{n-1}]_0^\infty + \int_0^\infty (n-1)\, e^{-x} x^{n-2} dx$$

$$= (n-1) \int_0^\infty e^{-x} x^{n-2} dx = (n-1)\,\Gamma(n-1)$$

Thus we have

$$\Gamma n = (n-1)\,\Gamma\ (n-1) \quad \text{............... } (2)$$

Similarly , $\Gamma\ (n-1) = (n-2)\,\Gamma\ (n-2)$

Hence it follows that $\Gamma\ n = (n-1)(n-2)\,\Gamma\ (n-2)$

If *n* is a positive integer, then proceeding as above repeatedly, we get

$$\Gamma\ n = (n-1)(n-2)\,(n-3).....3.2.1\ \Gamma\ 1$$

But $\Gamma\ 1 = \int_0^\infty e^{-x} dx = [-e^{-x}]_0^\infty = 1$ .... (3)

Hence when *n* is a positive integer

$$\Gamma\ n = (n-1)(n-2)\,(n-3).....3.2.1 = (n-1)! \quad \text{.............. } (4)$$

Thus, if *n* is positive integer, the for all values of *n*

$$\Gamma\ n = (n-1)\Gamma\ (n-1) = (n-1)! \quad \text{.................... } (5)$$

This is a *fundamental property* of the Gamma Functions. From this, we may write

$$\Gamma\ (n+1) = n\Gamma\ n \quad \text{i,e } \Gamma\ n = \frac{\Gamma\ (n+1)}{n} \quad \text{............... } (6)$$

Putting n=0, we get

$$\Gamma\ 0 = \infty \quad \text{........... } (7)$$

It can be further shown that

$$\Gamma\ (-n) = \infty \quad \text{........... } (8)$$

$$\Gamma\ n = \frac{\Gamma\ (n+1)}{n} \Rightarrow \Gamma\ (-1) = \frac{\Gamma\ (-1+1)}{-1} = \frac{\Gamma\ 0}{-1} = -\ \Gamma\ 0 = \infty \text{ and so on}$$

Transformation of Gamma Function (Other Forms of Gamma Function)

By definition $\Gamma\ n = \int_0^\infty e^{-x} x^{n-1} dx$ .... (1)

(a) Substituting x = λy ∴ dx= λ dy in equation (1), we get

$$\Gamma\ n = \int_0^\infty e^{-\lambda y}\ \lambda^{n-1} y^{n-1}.\ \lambda\, dy = \lambda^n \int_0^\infty e^{-\lambda y}\ y^{n-1} dy \quad .... (2a)$$

$$\therefore \quad \int_0^\infty e^{-\lambda y}\ y^{n-1} dy = \frac{\Gamma n}{\lambda^n} \quad .... (2b)$$

(b) Substituting $e^{-x} = y$, ∴ $x = log_e \frac{1}{y}$ and $dx = -\frac{dy}{y}$ in eq(1), we get

$$\Gamma\ n = \int_0^\infty e^{-x} x^{n-1} dx = -\int_1^0 y(log\ \frac{1}{y})^{n-1} \frac{dy}{y} = \int_0^1 (log\ \frac{1}{y})^{n-1} dy \quad .... (3)$$

(c) substituting $x^n = y$ ∴ $x = y^{1/n}$ and $dx = \frac{1}{n}\ y^{(1-n)/n} dy$ in equation (1), we get

$$\Gamma\ n = \int_0^\infty e^{-x} x^{n-1} dx = \int_0^\infty e^{-y^{1/n}}\ y^{(n-1)/n}\ \frac{1}{n}\ y^{(1-n)/n} dy$$

$$= \frac{1}{n} \int_0^\infty e^{-y^{1/n}} dy \quad .... (4)$$

Equations (2a), (3) and (4) represent transformed (other forms of Gamma Function)

Corollary. From eq (4), we have

$$\Gamma\ n = \frac{1}{n} \int_0^\infty e^{-y^{1/n}} dy$$

$$n.\ \Gamma\ n = \int_0^\infty e^{-y^{1/n}} dy \quad .... (5)$$

Replacing n by $\frac{1}{2}$ in equation (5), we get

$$\frac{1}{2}.\ \Gamma\ \frac{1}{2} = \int_0^\infty e^{-y^2} dy = \frac{\sqrt{\pi}}{2} \quad \text{(since } \int_0^\infty e^{-y^2} dy = \frac{\sqrt{\pi}}{2}\text{, refer section 4.8a)}$$

$$\Gamma\ \frac{1}{2} = \sqrt{\pi} \quad .... (6)$$

Relation between Beta and Gamma Functions

$$\beta(m,\, n) \;=\; \frac{\Gamma m\ \Gamma\ n}{\Gamma(m+n)}$$

The transformed form (2a) of section 4.6 of Gamma Function Γm is given by

$$\Gamma m \;=\; \int_0^\infty e^{-\lambda x}\, \lambda^m\, x^{m-1}\, dx \qquad \text{.................} \; (1)$$

Multiplying both sides by $e^{-\lambda}\, \lambda^{n-1}$ and integrating with respect to λ between the limits 0 and ∞, we get

$$\Gamma m \int_0^\infty e^{-\lambda}\, \lambda^{n-1}\, d\lambda \;=\; \int_0^\infty \Big[\int_0^\infty e^{-\lambda(1+x)}.\ \lambda^{m+n-1} d\lambda\Big]\, x^{m-1} dx \quad \text{............} \; (2)$$

But $\quad \int_0^\infty e^{-\lambda}\, \lambda^{n-1}\, d\lambda \;=\; \Gamma n \quad$ and also here x=λ

$$\int_0^\infty e^{-\lambda(1+x)}.\ \lambda^{m+n-1} d\lambda \;=\; \frac{\Gamma(m+n)}{(1+x)^{m+n}}$$ by (2b) of section 4.6

$$\int_0^\infty e^{-\lambda y}\, y^{n-1} dy \;=\; \frac{\Gamma n}{\lambda^n}$$ here (1+x)= λ, n=m+n and y=λ

Substituting these values in eq (2), we get

$$\Gamma m \Gamma n \;=\; \int_0^\infty \frac{\Gamma(m+n)}{(1+x)^{m+n}}\, x^{m-1} dx \qquad \text{.........} \; (3a)$$

$$=\; \Gamma(m\,+\,n) \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

$$from\ equation\ (2)\ of\ section\ 4.4\ is \quad \beta(m,\, n) \;=\; \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dx$$

$=\; \Gamma(m\,+\,n)\ \beta(m,\, n)$ by equation (2) of section 4.4

$$\therefore \quad \beta(m,\, n) \;=\; \frac{\Gamma m\, \Gamma n}{\Gamma(m+n)} \qquad \text{...........} \; (3b)$$

This is an important relation between Beta and Gamma functions.
