Mechanics of Particles

Newton's First Law: "Every body continues in the state of rest or uniform motion in a straight line unless an external force acts on it to change that state".

This law introduces "Inertia". This is the inherent property of a particle to oppose the change of state. It can be measured with the mass of a particle.

Newton's Second Law of Motion: "The rate of change of momentum of a body is equal to the net force acting on the body and takes place in the direction of force.

A particle of mass 'm' moving with initial velocity 'u'. Due to force 'F' applied for t sec, the final velocity is v.

Initial momentum P_i = mu

Final momentum P_f = mv

Change in momentum $P_f - P_i$ = mv-mu=m(v-u)

Rate of change of momentum = $\frac{m(v-u)}{t}$ = ma (acceleration a = $\frac{(v-u)}{t}$)

From Newton's second law F= ma

Newton's third law of motion: This law explains mutual interacting forces among the particles.

"To every action, there is always an equal and opposite reaction".

System of Variable Mass: In classical mechanics the mass of a system is always a constant such systems are isolated systems. For systems whose mass varies continuously over a period of time are variable mass systems.

Ex: Motion of a Rocket. Most of the mass of a rocket is its fuel. Fuel burns and gasses come out of the rocket with high speed through the nozzle. Rocket moves in the opposite direction to the direction of gasses.

At time t let rocket mass is M with velocity V. After Δt, let mass ΔM ejected out through a nozzle with velocity u. External force acting on the rocket is F_{ext}.

At a time t, initial momentum P_1 = MV

At a time (t+Δt) final momentum $P_2 = (M-\Delta M)(V+\Delta V) + \Delta M u$

ΔV is an increment in velocity of the rocket due to loss in mass in the time interval Δt.

$$F_{ext} = \frac{\Delta P}{\Delta t} = \frac{P_2 - P_1}{\Delta t} = \frac{(M-\Delta M)(V+\Delta V)+\Delta Mu - MV}{\Delta t} = \frac{M\Delta V}{\Delta t} + \frac{\Delta M}{\Delta t}((-V - \Delta V + u)$$

$$= \frac{M\Delta V}{\Delta t} + \frac{\Delta M}{\Delta t}(u - (V + \Delta V))$$

$\Delta M = -\Delta M$ as mass decreases

$F_{ext} = \frac{M\Delta V}{\Delta t} - \frac{\Delta M}{\Delta t}(u - (V + \Delta V))$

As $\Delta t \rightarrow 0$, $\Delta V \rightarrow 0$

$F_{ext} = \frac{MdV}{dt} - \frac{dM}{dt}(u - V) = \frac{MdV}{dt} - u\frac{dM}{dt} + V\frac{dM}{dt} = \frac{d}{dt}(MV) - u\frac{dM}{dt}$

Newton's equation of motion for a variable mass system

Case1: For a system with constant Mass $\frac{dM}{dt} = 0$

$F_{ext} = \frac{d}{dt}(MV) = M\frac{dV}{dt} = ma$ This is Newton's second law

Case 2: From $F_{ext} = \frac{MdV}{dt} - \frac{dM}{dt}(u - V)$

$$\frac{MdV}{dt} = F_{ext} + \frac{dM}{dt}(u - V)$$

$\frac{MdV}{dt} = F_{ext} + u_{rel}\frac{dM}{dt}$

$$\frac{MdV}{dt} = instantaneous\, force\, acting\, on\, rocket$$

$u_{rel}\frac{dM}{dt}$ = force of thrust known as reaction of force

$\frac{MdV}{dt} = F_{ext} + F_{reaction}$

Motion of a Rocket: The fuel in the rocket produces a high pressure within the chamber after burning. The burnt gasses eject out of the rocket with high velocity. This is action. Rocket moves in the opposite direction. This is a reaction. Hence the motion depends on Newton's third law.

At a time 't' sec, Mass of a rocket = M, Velocity = V, Rate of change of mass = $\frac{dM}{dt}$

$$F_{reaction} = -\frac{dM}{dt}u_{rel}$$

Due to this force, the rocket moves upward with certain acceleration.

Case I: If no external forces are acting on the rocket

$\frac{MdV}{dt} = -u_{rel}\frac{dM}{dt}$

a= $\frac{dV}{dt} = \frac{u_{rel}\frac{dM}{dt}}{M}$

Acceleration of a rocket

1) Proportional to u_{rel} (relative velocity)
2) Proportional to rate of change of mass $\frac{dM}{dt}$
3) Proportional to 1/M

$$dV = -u_{rel}\frac{dM}{M}$$

$$\int dV = -\int u_{rel}\frac{dM}{dt} = -u_{rel}\int\frac{dM}{dt}$$

$$V = -u_{rel}\log M + C$$

Initially mass of rocket = M_0, initial velocity = V_0

$$V_0 = -\log M_0 + C$$

$$C = V_0 + u_{rel}\log M_0$$

$$V = -u_{rel}\log M + V_0 + u_{rel}\log M_0 \quad = V_0 + u_{rel}\log\frac{M_0}{M}$$

This equation gives the velocity of the rocket at any instant of time.

This is known as the first equation of motion of a rocket.

Case II: If the external force is acting on the rocket

$$\frac{MdV}{dt} = -u_{rel}\frac{dM}{dt} - Mg$$

$$dV = -u_{rel}\frac{dM}{dt} - gdt$$

Integrating on both sides, we have

$$\int dV = -\int u_{rel}\frac{dM}{dt} - \int gdt$$

$$V = -u_{rel}\log M - gt + C$$

At t= 0, $M=M_0$, $V = V_0$

$$V_0 = -u_{rel}\log M_0 + C$$

$$V = -u_{rel}\log M - gdt + V_0 + u_{rel}\log M_0 \quad = V_0 + u_{rel}\log\frac{M_0}{M} - gt$$

This is known as the second equation of motion of a rocket.

Multistage rocket:

To obtain maximum velocity for a rocket at final stage

1. Relative velocity of gasses to be maximum
2. Final mass of the rocket M is very much less than initial mass of the rocket M_0

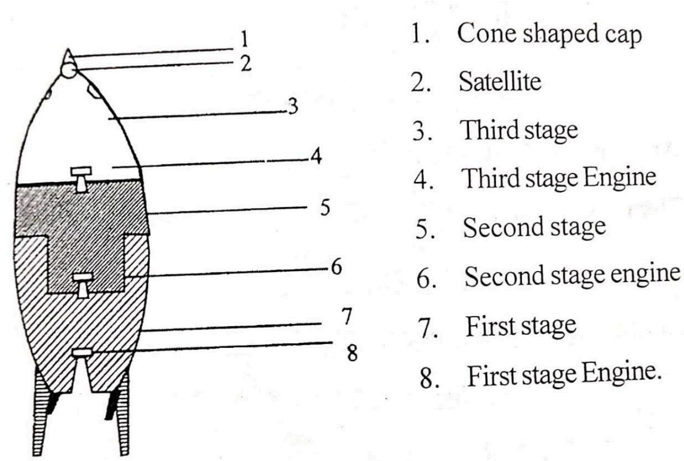
Relative velocity of gasses coming out of the rocket depends on temperature and pressure within the chamber. It also depends on the area of the cross section of the nozzle.

For optimum design of the fuel chamber $\frac{M_0}{M}$ is maintained at nearly 10.

Neglecting gravitational force

V = 0+2 $\log log\ 10$ -0 = 2 x 2.3 = 4.6 Km/sec

This is less than the orbital velocity of a rocket (I.e. 11.2 km/sec). Due to this reason, in order to launch a satellite, multistage rockets are used.



Multistage Rocket

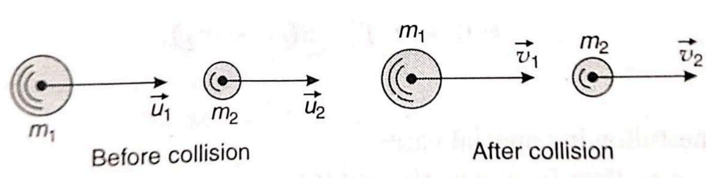
At the end of the first stage, the velocity of the rocket was nearly 4.6 km/sec. When the fuel of the first stage is exhausted, it detaches from the rocket and drops off.
The velocity at this stage becomes the initial velocity of the second stage. When the fuel of the second stage is exhausted, it too is detached from the rocket. Finally the third stage rocket starts off with the required velocity.

Collision: A redistribution of the total momentum of the total momentum of the bodies (or particles) when they approach each other.

Elastic collision: When the kinetic energy and momentum of the particles remains conserved in the collision, the collision is said to be elastic collision

Elastic collision in one dimension:

Let us consider an elastic one – dimensional collision between two particles. Let m_1 and m_2 be the masses of two particles. Suppose u_1, u_2 be their respective velocities before and after the collision.



Applying the law of conservation of linear momentum, we have

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

$m_1(u_1 - v_1) = m_2(v_2 - u_2)$----------------1

According to law of conservation of kinetic energy, we have

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

$$m_1(u_1^2 - v_1^2) = m_2(v_2^2 - u_2^2) \text{ -------------2}$$

Dividing equation 2 by equation 1

$$\frac{m_1(u_1^2-v_1^2)= m_2(v_2^2-u_2^2)}{m_1(u_1-v_1) =m_2(v_2-u_2)}$$

$$\frac{(u_1-v_1)(u_1+v_1)= (v_2-u_2)(v_2+u_2)}{(u_1-v_1) =(v_2-u_2)}$$

$$u_1 + v_1 = v_2 + u_2$$

$$v_2 = u_1 + v_1 - u_2 \text{----------3}$$

Substituting this value of v_2 in equation 1, we get

$$m_1(u_1 - v_1) = m_2((u_1 + v_1 - u_2) - u_2)$$

$$m_1u_1 - m_1v_1 = m_2u_1 + m_2v_1 - 2m_2u_2$$

$$m_1u_1 - m_2u_1 + 2m_2u_2 = m_2v_1 + m_1v_1$$

$$u_1\left(m_1 - m_2\right) + 2m_2u_2 = (m_2 + m_1)v_1$$

$$v_1 = \frac{\left(m_1-m_2\right)u_1}{(m_1+m_2)} + \frac{2m_2u_2}{(m_1+m_2)}$$

From equation 3

$$v_1 = v_2 + u_2 - u_1$$

$$m_1(u_1 - (v_2 + u_2 - u_1) = m_2(v_2 - u_2)$$

$$m_1u_1 - m_1v_2 - m_1u_2 + m_1u_1 = m_2v_2 - m_2u_2$$

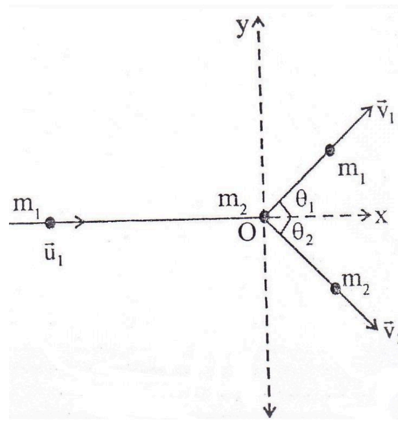
$$2m_1u_1 - m_1u_2 + m_2u_2 = m_2v_2 + m_1v_2$$

$$2m_1u_1 + \left(m_2 - m_1\right)u_2 = (m_2 + m_1)v_2$$

$$v_2 = \frac{2m_1u_1}{(m_1+m_2)} + \frac{\left(m_2-m_1\right)u_2}{(m_1+m_2)}$$

Elastic collision in two dimensions:

Let a particle of mass m_1 moving with velocity u_1 collide with a particle m_2 at rest ($u_2 = 0$). Let, after collision, the particle of mass m_1 be deflected or scattered at an angle θ_1 with original direction. Similarly, the particle of mass m_2 moves in a direction which makes an angle θ_2 with the original direction. Further let v_1 *and* v_2 be the velocities of masses m_1 *and* m_2 respectively after collision.



Applying the law of conservation of linear momentum along x-axis, we have

$m_1u_1 + 0 = m_1v_1cos\theta_1 + m_2v_2cos\theta_2$ ....(1)

Applying the law of conservation of linear momentum along Y-axis, we have

$$0 + 0 = m_1v_1sin\theta_1 - m_2v_2sin\theta_2$$

$m_1v_1sin\theta_1 = m_2v_2sin\theta_2$ ....(2)

According to the law of conservation of kinetic energy

$\frac{1}{2}m_1u_1^2 + 0 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$ ....(3)

If $m_1 = m_2$

From equation (1) $u_1 + 0 = v_1cos\theta_1 + v_2cos\theta_2$ ………………..(4)

From equation (2) $v_1sin\theta_1 = v_2sin\theta_2$ ………………………..(5)

From equation (3) $u_1^2 = v_1^2 + v_2^2$ .... (6)

From equation (4) $v_2cos\theta_2 = u_1 - v_1cos\theta_1$

Squaring both sides, we get $(v_2cos\theta_2)^2 = (u_1 - v_1cos\theta_1)^2$

$v_2^2cos^2\theta_2 = u_1^2 + v_1^2cos^2\theta_1 - 2u_1v_1cos\theta_1$.................(7)

Squaring both sides of equation (5), we have

$(v_2sin\theta_2)^2 = (v_1sin\theta_1)^2$ ....(8)

Adding equations (7) and (8), we get

$$v_2^2cos^2\theta_2 + v_2^2sin^2\theta_2 = u_1^2 + v_1^2cos^2\theta_1 - 2u_1v_1cos\theta_1 + v_1^2sin^2\theta_1$$

$$v_2^2(\cos^2\theta_2 + \sin^2\theta_2) = u_1^2 + v_1^2(\cos^2\theta_1 + \sin^2\theta_1) - 2u_1v_1\cos\theta_1$$

$v_2^2 = u_1^2 + v_1^2 - 2u_1v_1\cos\theta_1$…………………(9)

From equation (6) $v_2^2 = u_1^2 - v_1^2$ ………………………………(10)

From equations (9) and (10)

$$u_1^2 - v_1^2 = u_1^2 + v_1^2 - 2u_1v_1\cos\theta_1$$

$$2v_1^2 = 2u_1v_1\cos\theta_1$$

$v_1 = u_1\cos\theta_1$…………………………(11)

From equation (10) $v_2^2 = u_1^2 - v_1^2$

$v_2^2 = u_1^2 - u_1^2\cos^2\theta_1 = u_1^2\left(1 - \cos^2\theta_1\right) = u_1^2\sin^2\theta_1$

$v_2 = u_1\sin\theta_1$……………………(12)

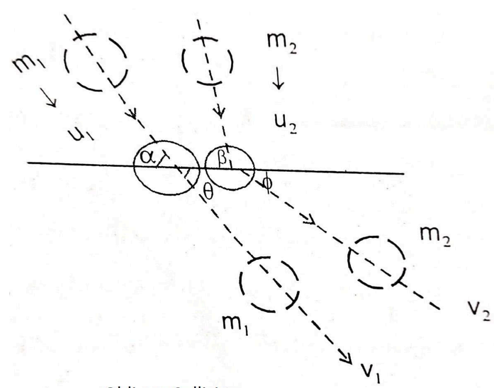
From equations (11) and (12), it is clear that v_1 and v_2 are perpendicular components of u_1

$$\theta_1 + \theta_2 = 90^0$$

In a perfectly elastic collision between particles of equal masses, when one particle is initially at rest, the two particles always move off at right angles to each other after collision.

Elastic oblique collision:

Let two bodies of masses m_1 and m_2 be moving in different directions with velocities u_1 and u_2 respectively collide with each other. After collision, the two bodies move in different directions with different velocities.



Oblique Collision

Applying the law of conservation of linear momentum, we have

$$m_1u_1\cos\alpha + m_2u_2\cos\beta = m_1v_1\cos\theta + m_2v_2\cos\emptyset$$

$m_1(u_1\cos\alpha - v_1\cos\theta) = m_2(v_2\cos\emptyset - u_2\cos\beta)$----------------1

According to law of conservation of kinetic energy, we have

$$\tfrac{1}{2}m_1u_1^2\cos^2\alpha + \tfrac{1}{2}m_2u_2^2\cos^2\beta = \tfrac{1}{2}m_1v_1^2\cos^2\theta + \tfrac{1}{2}m_2v_2^2\cos^2\emptyset$$

$m_1(u_1^2cos^2\alpha - v_1^2cos^2\theta) = m_2(v_2^2cos^2\emptyset - u_2^2cos^2\beta)$ -------------2

Dividing equation 2 by equation 1

$$\frac{m_1(u_1^2cos^2\alpha - v_1^2cos^2\theta) = m_2(v_2^2cos^2\emptyset - u_2^2cos^2\beta)}{m_1(u_1cos\,\alpha - v_1cos^2\theta) = m_2(v_2cos\,\emptyset - u_2cos\,\beta)}$$

$$\frac{(u_1cos\,\alpha - v_1cos\,\theta)(u_1cos\,\alpha + v_1cos\,\theta) = (v_2cos\,\emptyset - u_2cos\,\beta)(v_2cos\,\emptyset + u_2cos\,\beta)}{(u_1cos\,\alpha - v_1cos\,\theta) = (v_2cos\,\emptyset - u_2cos\,\beta)}$$

$$u_1cos\,\alpha + v_1cos\,\theta = v_2cos\,\emptyset + u_2cos\,\beta$$

$v_2cos\,\emptyset = u_1cos\,\alpha + v_1cos\,\theta - u_2cos\,\beta$ ----------3

Substituting this value of v_2 in equation 1, we get

$$m_1(u_1cos\,\alpha - v_1cos\,\theta) = m_2((u_1cos\,\alpha + v_1cos\,\theta - u_2cos\,\beta) - u_2cos\,\beta)$$

$$m_1u_1cos\,\alpha - m_1v_1cos\,\theta = m_2u_1cos\,\alpha + m_2v_1cos\,\theta - 2m_2u_2cos\,\beta$$

$$m_1u_1cos\,\alpha - m_2u_1cos\,\alpha + 2m_2u_2cos\,\beta = m_2v_1cos\,\theta + m_1v_1cos\,\theta$$

$$u_1cos\,\alpha\left(m_1 - m_2\right) + 2m_2u_2cos\,\beta = (m_2 + m_1)v_1cos\,\theta$$

$$v_1cos\,\theta = \frac{\left(m_1 - m_2\right)u_1cos\,\alpha}{(m_1 + m_2)} + \frac{2m_2u_2cos\,\beta}{(m_1 + m_2)}$$

From equation 3

$$v_1cos\,\theta = v_2cos\,\emptyset + u_2cos\,\beta - u_1cos\,\alpha$$

$$m_1(u_1cos\,\alpha - (v_2cos\,\emptyset + u_2cos\,\beta - u_1cos\,\alpha) = m_2(v_2 - u_2cos\,\beta)$$

$$m_1u_1cos\,\alpha - m_1v_2cos\,\emptyset - m_1u_2cos\,\beta + m_1u_1cos\,\alpha = m_2v_2cos\,\emptyset - m_2u_2cos\,\beta$$

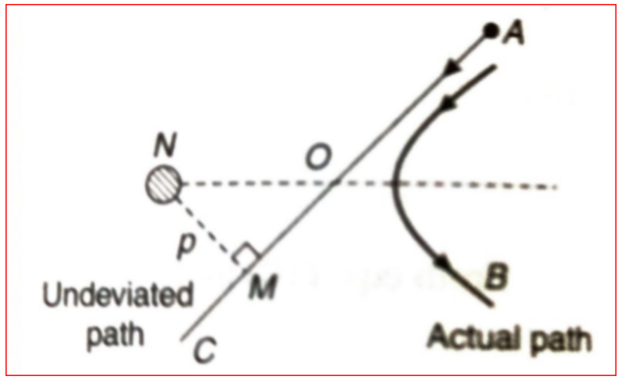
$$2m_1u_1cos\,\alpha - m_1u_2cos\,\beta + m_2u_2cos\,\beta = m_2v_2cos\,\emptyset + m_1v_2cos\,\emptyset$$

$$2m_1u_1cos\,\alpha + \left(m_2 - m_1\right)u_2cos\,\beta = (m_2 + m_1)v_2cos\,\emptyset$$

$$v_2cos\,\emptyset = \frac{2m_1u_1cos\,\alpha}{(m_1 + m_2)} + \frac{\left(m_2 - m_1\right)u_2cos\,\beta}{(m_1 + m_2)}$$

Impact parameter:

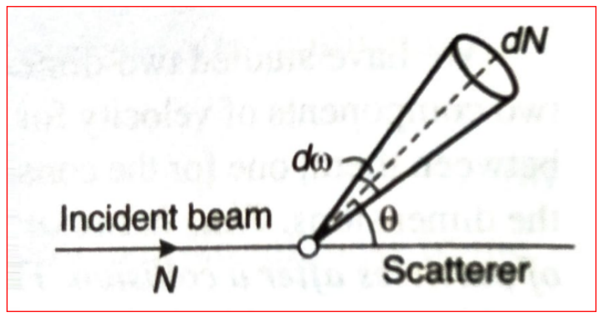
Consider a positive particle, like a proton or an α-particle, approaching a massive nucleus N of an atom. Due to the Coulombic force of repulsion, the particle follows a hyperbolic path AB with nucleus N as its focus. In the absence of the repulsive force, the particle would have followed the straight line path AC. p is the perpendicular distance from the nucleus N to the original direction AC of the particle. The distance (NM=p) is called the impact parameter.



"The closest distance between the nucleus and positively charged particle projected towards it. This is also known as the `` collision parameter".

Scattering cross – section:

When α-particles are projected into a thin metal foil, they are deflected or scattered in different directions. Let N be the incident intensity (number of incident particles crossing per unit time a unit surface placed perpendicular to the direction of propagation). Suppose dN be the number of particles scattered per unit time into solid angle dω located in the direction θ and φ with respect to the bombarding direction. The ratio dN/N is called scattering cross – section.



"The scattering cross-section in a given direction is defined as the ratio of number of scattering particles into solid angle dω per unit time to the incident intensity.

Scattering cross – section $\sigma_{sc} = \frac{dN}{N}$