Central Forces

Central Force: A force which always acts on a particle towards or away from a fixed point and magnitude depends only on the distance from the fixed point is called "Central Force".

Examples:

1. The earth moves around the sun under central force which is always directed towards the sun.
2. The electrostatic force of attraction or repulsion between two point charges is a central force.
3. The elastic force acting on a mass attached to one end of a spring is a central force.

Let us consider a particle P whose polar coordinates $\bar{r}$ and θ. The central force can be expressed as $\bar{F} = \hat{r} f(r)$

Here f(r) is a function of the distance r and $\hat{r}$ is unit vector along the radius vector $\bar{r}$.

Example of central forces:

1) Gravitational force:

The gravitational attraction force between masses m_1,m_2 separated at a distance r is given by $\bar{F}_{12} = -\frac{Gm_1m_2}{r^2}\hat{r}$

Here negative represent attractive force.

From above equation, Force can be expressed as

$\bar{F} = \hat{r} f(r)$ where f(r) = - $\frac{Gm_1m_2}{r^2}$

2) Electrostatic force:
   Electrostatic force between two charges q_1 and q_2 separated a distance r is given by
   $F_{12} = \frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{r^2}\hat{r}$
   Let f(r) = $\frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{r^2}$
   Then it can be expressed as F_{12} = f(r) $\hat{r}$
3) The elastic force acting on a mass suspended from a spring is expressed as
   F = - kx
   This is acting towards a fixed point. So, it is a central force.

Features of the central forces:

1) Forces acting towards or away from a fixed point.
2) Central force is conservative force.
3) Under central force, the angular momentum of the particle is conserved.
4) Under central force, the torque acting on the particle is zero.
5) Under central force, the areal velocity of the particle is constant.
6) The central force is attractive, if f(r) <0 and repulsive if f(r)>0.

Conservative nature of central force:

A force is said to be conservative when work done by the force to move the particle from one point to another point is independent of path followed.

A particle is to be moved from point A to point B along paths I and II by applying conservative like central force F.

The work done by the central force in moving the particle from point A to B along the paths I is

W(A→B) = $\int_A^B F.dr$ ------(1)

The work done by force along path II from B to A is given by

W(B→A) = $\int_B^A F.dr$ = - $\int_A^B F.dr$

Or $\int_A^B F.dr = -w$(B→A) ----(2)

From eqn (1) & (2)

W(A→B) = - W(B→A)

W(A→B) + W(B→A) = 0

Conservative force as a negative gradient of potential energy:

Consider a conservative force F moves the particle from point (x_0,y_0,z_0) to another point (x,y,z) in the space. Then potential energy U be expressed in terms of conservative force is given by

U (r) = - $\int_{r_0}^{r} F.dr$ -----(1)

r_0, r are position vectors of points (x_0,y_0,z_0) and (x,y,z)

Let $F = iF_x + jF_y + kF_z$ and $r = xi + yj + zk$

$Dr = dxi + dyj + dzk$

Then $F.dr = (iF_x + jF_y + kF_z).(dxi + dyj + dzk)$

$F.dr = F_x dx + F_y dy + F_z dz$

Substituting this value in eqn (1) we get

$U(r) = -\int_{r_0}^{r} F.dr = -\int_{x_0}^{x} F_x.dx - \int_{y_0}^{y} F_y.dy - \int_{z_0}^{z} F_z.dz$

Differentiating the eqn partially with respect to x,y,z we get

$F_x = -\frac{\partial U}{\partial x}$, $F_y = -\frac{\partial U}{\partial y}$, $F_z = -\frac{\partial U}{\partial z}$

Now $F = F_x i + F_y j + F_z K$

$= -\frac{\partial U}{\partial x} i - \frac{\partial U}{\partial y} j - \frac{\partial U}{\partial z} k$

$= -(\frac{\partial U}{\partial x} i + \frac{\partial U}{\partial y} j + \frac{\partial U}{\partial z} k)$

$= -\nabla U$

Conservative Force is negative gradient of potential energy.

Radial and Centripetal acceleration in polar coordinates

Consider the plane polar coordinates $\bar{r}$ and θ for the position of a particle p. Let $\hat{r}, \hat{\theta}$ be the unit vectors along and perpendicular to radius vector r.

$\bar{r} = r\ \hat{r}$

$\frac{d\hat{r}}{dt} = \frac{d\theta}{dt}.\hat{\theta}$

Velocity $v = \frac{dr}{dt} = \frac{dr}{dt}.\hat{r} + r\ \frac{d\hat{r}}{dt}$

$\frac{d\hat{\theta}}{dt} = -\frac{d\theta}{dt}.\hat{r}$

$= \frac{dr}{dt}.\hat{r} + r.\ \frac{d\theta}{dt}.\hat{\theta}$

In the above equation, $\frac{dr}{dt}$ is the radial component of the velocity of the particle. This is due to change in magnitude of r when θ is constant. $r.\ \frac{d\theta}{dt}$ is transverse component of velocity of the particle. This is due to the change in θ when r is constant.

Acceleration $a = \frac{d^2r}{dt^2} = \frac{d}{dt}(\frac{dr}{dt}.\hat{r} + r.\ \frac{d\theta}{dt}.\hat{\theta})$

$$= \frac{d^2r}{dt^2}.\hat{r} + \frac{dr}{dt}.\ \frac{d\hat{r}}{dt} + \frac{dr}{dt}.\ \frac{d\theta}{dt}.\hat{\theta} + r.\frac{d^2\theta}{dt^2}.\hat{\theta} + r.\frac{d\theta}{dt}.\frac{d\hat{\theta}}{dt})$$

$$= \frac{d^2r}{dt^2}.\hat{r} + \frac{dr}{dt}.\ \frac{d\theta}{dt}.\hat{\theta} + \frac{dr}{dt}.\ \frac{d\theta}{dt}.\hat{\theta} + r.\frac{d^2\theta}{dt^2}.\hat{\theta} - r.\frac{d\theta}{dt}.\frac{d\theta}{dt}.\hat{r}$$

$$= \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{r} + \left[r.\frac{d^2\theta}{dt^2} + 2.\frac{dr}{dt}.\ \frac{d\theta}{dt}\right]\hat{\theta}$$

Radial component of acceleration $= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$

Transverse component of acceleration = $r.\frac{d^2\theta}{dt^2} + 2.\frac{dr}{dt}.\ \frac{d\theta}{dt} = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$

Gravitational field and Gravitational potential

Gravitational field.

The space around a body within which its gravitational force of attraction can be experienced is called the gravitational field.

Gravitational attraction: Gravitational attraction or the intensity of the gravitational field at a point in the field is the force experienced by a unit mass placed at that point.

The intensity of the gravitational field at a point distant r cm from a point mass M.

f(r) = - $\frac{GM\ X\ 1}{r^2} = \frac{GM}{r^2}$ This vector quantity.

The intensity of gravitational field:

1) The gravitational field strength at any point varies inversely as the square of the distance from the mass M
2) The force due to M on any mass m can be calculated by multiplying $\frac{GM}{r^2}$ with m.
3) If the distribution of matter in space does not change with time, then the gravitational field at any point in space will also not change with time. i.e. gravitational field is stationary one.
4) The gravitational field is a vector field that acts along the lines joining the centres of the two bodies.

Gravitational potential:

The gravitational potential at a point in a gravitational field is the work done in taking a unit mass from infinity to that point.

The difference of gravitational potential between two points in a gravitational field is the work done in taking a unit mass from one point to the other against the gravitational force of attraction.

Gravitational potential due to a point mass:

Let M be an attracting point mass placed at O. If a unit mass is placed at A then gravitational attraction experienced by it is $\frac{GM}{r^2}$ which will be directed towards.

If the unit mass is moved against this attraction through a small distance dr to point B, the work done against the attraction is $\frac{GM}{r^2}\ X\ dr$

Hence work done in moving a unit mass from A to infinity is given by

$$= \int_r^{\infty} \frac{GM}{r^2}\ dr = GM\left[-\frac{1}{r}\right]_r^{\infty} = \frac{GM}{r}$$

The work done by the field in moving unit mass from infinity to A is $-\frac{GM}{r}$

The gravitational potential V = $-\frac{GM}{r}$

Equation of motion under a central force:

When a body moves under the action of a central force, the force is radial and is always towards a fixed point.

Radial acceleration $= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$ ….(1)

The transverse acceleration of the body is zero, as there is no force acting on the particle perpendicular to r, then

Transverse acceleration = $r.\frac{d^2\theta}{dt^2} + 2.\frac{dr}{dt}.\ \frac{d\theta}{dt} = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0$ …(2)

$$r^2\frac{d\theta}{dt} = h = \ constant$$

Let us take $\frac{1}{r}$ = u or $r = \frac{1}{u}$

Derivation above expression with respect to t .

$$\frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}.\frac{d\theta}{dt}$$

$$= -\left(r^2\frac{d\theta}{dt}\right).\frac{du}{d\theta} = -h\frac{du}{d\theta} \quad where\ h = r^2\frac{d\theta}{dt} \text{ or } \frac{d\theta}{dt} = \frac{h}{r^2}$$

Deriving again with respect to t, we get

$$\frac{d^2r}{dt^2} = \frac{d}{dt}\left(-h\frac{du}{d\theta}\right) = -h\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)\frac{d\theta}{dt}$$

$$= -h\frac{d^2u}{d\theta^2}.\frac{d\theta}{dt} = -h\frac{d^2u}{d\theta^2}.\frac{h}{r^2} = -h^2u^2\frac{d^2u}{d\theta^2}$$

Substituting value of $\frac{d^2r}{dt^2}$ in eqn (1)

Radial acceleration = $\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -h^2u^2\frac{d^2u}{d\theta^2} - r\left(\frac{d\theta}{dt}\right)^2$

$$= -h^2u^2\frac{d^2u}{d\theta^2} - r\frac{h^2}{r^4}$$

$$= -h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3$$

Now force acting on the particle = mass x radial acceleration

$$F = -m[-h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3]$$

Here negative sign indicate attractive force.

$$F = m[h^2u^2\frac{d^2u}{d\theta^2} + h^2u^3]$$

Let $\frac{F}{m} = P, then$

$$P = h^2u^2\frac{d^2u}{d\theta^2} + h^2u^3 = h^2u^2(\frac{d^2u}{d\theta^2} + u)$$

$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2}$

Kepler's laws:

First law: The path of a planet is an elliptical orbit around the sun, with sun at one of its foci.

Second law: The radius vector, drawn from the sun to a planet, sweeps out equal areas in equal time. i.e. its areal velocity is constant.

Third law: The square of time period of the planet round the sun is proportional to the cube of the semi – major axis of its orbit.

Deduction of Kepler's laws:

First law:

Consider the case of a planet of mass m rotating round the sun of mass M in an orbit of radius r. According to Newton's law of gravitation

The force of attraction F = $G\frac{mM}{r^2}$ This force is directed towards the centre of the sun.

Force for unit mass P = $\frac{F}{m}$ = $G\frac{M}{r^2}$ = μu^2 (where GM = μ and 1/r = u)

The polar equation of central orbit is $\frac{d^2u}{d\theta^2} + u = \frac{p}{h^2u^2}$ or $\frac{d^2u}{d\theta^2} + u = \frac{\mu u^2}{h^2u^2}$

$$\frac{d^2u}{d\theta^2} + \left(u - \frac{\mu}{h^2}\right) = 0 \;\; or \;\; \frac{d^2}{d\theta^2}\left(u - \frac{\mu}{h^2}\right) + \left(u - \frac{\mu}{h^2}\right) = 0$$

$\frac{d^2x}{d\theta^2} + x = 0$ Where $\left(u - \frac{\mu}{h^2}\right) = x$

This is second degree differential equation whose solution is

x = A cos (θ-θ_0), where A and θ_0 are constants.

Putting the value of x, we get

$$u - \frac{\mu}{h^2} = A\cos(\theta - \theta_0) \;or\; u = \frac{\mu}{h^2} + A\cos(\theta - \theta_0)$$

$$\frac{1}{r} = \frac{\mu}{h^2} + A\cos(\theta - \theta_0)$$

$$\frac{1}{r} = \frac{1 + Ah^2/\mu\cos(\theta - \theta_0)}{(h^2/\mu)}$$

This is similar to polar equation of a conic.

$$\frac{1}{r} = \frac{1 + \varepsilon\cos\theta}{l}$$

l= semi – latus rectum = h^2/μ

ε= eccentricity of the conic = $\frac{Ah^2}{\mu}$

ε>1, the conic is hyperbola

ε=1, the conic is parabola

ε<1, the conic is ellipse

So, in order to consider the nature of the conic, we have to find the value of ε

The kinetic energy K. E. = $\frac{1}{2}\left\{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right\}$

$= \frac{1}{2}m\left\{h^2\left(\frac{du}{d\theta}\right)^2 + h^2u^2\right\} = \frac{1}{2}mh^2\left\{A^2sin^2(\theta-\theta_0) + \left[\frac{\mu}{h^2} + A\cos(\theta-\theta_0)\right]^2\right\}$

$= \frac{1}{2}mh^2\left\{A^2sin^2(\theta-\theta_0) + \frac{\mu^2}{h^4} + A^2cos^2(\theta-\theta_0) + \frac{2\mu}{h^2}\ A\cos(\theta-\theta_0)\right\}$

$= \frac{1}{2}mh^2\left\{A^2 + \frac{\mu^2}{h^4} + \frac{2\mu}{h^2}\ A\cos(\theta-\theta_0)\right\}$

The potential energy of the planet is given by

P.E. $= \int_r^\infty -\frac{GmM}{r^2}dr = -\int_r^\infty \frac{\mu m}{r^2}dr = (\frac{\mu m}{r})_r^\infty = -\frac{\mu m}{r} = -\mu m u = -\mu m\left[\frac{\mu}{h^2} + A\cos(\theta-\theta_0)\right]$

Total energy of planet E = P.E. + K.E.

E $= \frac{1}{2}mh^2\left\{A^2 + \frac{\mu^2}{h^4} + \frac{2\mu}{h^2}\ A\cos(\theta-\theta_0)\right\} - \mu m\left[\frac{\mu}{h^2} + A\cos(\theta-\theta_0)\right]$

$$= \frac{1}{2}mh^2\left\{A^2 + \frac{\mu^2}{h^4} + \frac{2\mu}{h^2}\ A\cos(\theta-\theta_0) - \frac{2\mu^2}{h^4} - \frac{2\mu}{h^2}\ A\cos(\theta-\theta_0)\right\} = \frac{1}{2}mh^2\left\{A^2 - \frac{\mu^2}{h^4}\right\}$$

$$\frac{2E}{mh^2} = A^2 - \frac{\mu^2}{h^4} \quad or\ A^2 = \frac{2E}{mh^2} + \frac{\mu^2}{h^4}$$

$A = \left[\frac{2E}{mh^2} + \frac{\mu^2}{h^4}\right]^{1/2} = \frac{\mu}{h^2}\left[1 + \frac{2Eh^2}{m\mu^2}\right]^{1/2}$

$The\ eccentricity\ of\ the\ conic\ \ \varepsilon = \frac{Ah^2}{\mu} = \left[1 + \frac{2Eh^2}{m\mu^2}\right]^{1/2}$

This equation shows the orbits with different energies

If ε<1 or E<0, the orbit is ellipse

If ε=1 or E=0, the orbit is parabola

If ε>1 or E>0, the orbit is hyperbola.

A body with a parabola or hyperbola orbit would pass away from the solar system and will never return to it. i.e. the system is unbounded. Thus we conclude that the planetary orbits are all elliptical. i.e. bounded. This leads to Kepler's first law which states that every planet revolves round the sun in an elliptical orbit with sun at one of its foci.

Second law:

Let the planet move from P to P^1 in dt seconds. Now area swept by the radius vector in time dt is given by

dA $= \frac{1}{2}r^2d\theta \quad (area\ of\ triangle = \frac{1}{2}\ X\ base\ X\ height)$

Areal velocity $\frac{dA}{dt} = \frac{1}{2}r^2\left(\frac{d\theta}{dt}\right)$

$$but\ r^2\left(\frac{d\theta}{dt}\right) = h = constant$$

$\frac{dA}{dt} = \frac{h}{2} = constant$

This shows that areal velocity of the planet is constant. i.e. when a planet revolves round the sun, the radius vector joining the sun to the planet sweeps equal areas in equal time intervals.

Third law:

The time period of revolution T = $\frac{area\ swept\ in\ one\ revolution}{areal\ velocity} = \frac{\pi ab}{h/2} = \frac{2\pi ab}{h}$

Where πab = area of the ellipse with a and b as semi- major and semi -minor axes.

We know that semi – latus rectum $\frac{b^2}{a} = \frac{h^2}{\mu}\ or\ h^2 = \frac{\mu b^2}{a}\quad or\ h = b\sqrt{\left(\frac{\mu}{a}\right)}$

T = $\frac{2\pi ab}{h} = \frac{2\pi ab}{b\sqrt{\left(\frac{\mu}{a}\right)}} = \frac{2\pi a}{\sqrt{\left(\frac{\mu}{a}\right)}}$

$$T^2 = \frac{4\pi^2 a^2}{\mu/a} = \frac{4\pi^2 a^3}{\mu}$$
$$T^2 \ \alpha \ a^3$$

Thus the square of the period of revolution of a planet round the sun is proportional to the cube of semi – major axis.

Problems:

1. Show that the force F = (y2-x2)i + 2xy j is conservative

Solution: If the curl of the force is zero, then it is conservative.

$$Curl\ F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y^2 - x^2) & 2xy & 0 \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}.0 - \frac{\partial}{\partial z}(2xy)\right] + j\left[\frac{\partial}{\partial z}(y^2 - x^2) - \frac{\partial}{\partial x}.0\right] + k\left[\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(y^2 - x^2)\right].$$

$$= i[0-0] + j[0-0] + k[2y - 2y] = 0$$

$$Hence\ the\ force\ is\ conservative.$$

2. If the radius of the earth around the sun s doubled, find the new time period.

Solution: We know that $T^2 \propto a^3$

$\frac{T_1^2}{T_2^2} = \frac{a_1^3}{a_2^3}$ or $T_2 = T_1(\frac{a_2}{a_1})^{3/2}$

$$Given \quad T_1 = 1\ year\ and\ a_2 = 2a_1$$

or $T_2 = 1\ X\ (\frac{2a_1}{a_1})^{3/2} = 1\ X\ (2)^{3/2} = 2\sqrt{2} = 2.828\ years$

3. If the earth be one – half of its present distance from the sun, what will be the number of days in a year?

Solution: According to Kepler's law $T^2 \propto a^3$

Where T is the time period and 2a is the major axis of the orbital of the ellipse.

$\frac{T_1^2}{T_2^2} = \frac{a_1^3}{a_2^3}$

$$Here\ T_1 = 365\ days, \quad a_2 = \frac{1}{2}x, \quad a_1 = x\ (say) \quad and\ T_2 = ?$$

$$\left(\frac{365}{T_2}\right)^2 = \left(\frac{x}{x/2}\right)^3 = 8$$

$$T_2^2 = \frac{(365)^2}{8} = 129\ days$$

4. The Jupiter's period of revolution round the sun is 12 times that of the earth. Assuming the planetary orbitals to be circular find how many times the distance between the Jupiter and sun exceeds that between the earth and the sun.

Solution: we know that $T^2 \propto a^3$

$$\frac{T_1^2}{T_2^2} = \frac{a_1^3}{a_2^3}$$

$$period\ of\ revolution\ of\ earth\ T_2 = 12\ T$$

$$Jupiter's\ period\ of\ revolution\ T_1 = T$$

$radius\ of\ the\ earth\ a_2$,

$$radius\ of\ the\ Jupitor\ \ a_1$$

From above equation

$$\frac{T_1^2}{T_2^2} = \frac{a_1^3}{a_2^3}$$

$$\frac{a_1^3}{a_2^3} = \frac{T_1^2}{T_2^2} = \frac{144T^2}{T^2} = 144$$

$$\frac{a_1}{a_2} = (144)^{1/3} = 5.242$$

Hence the Jupiter's distance is 5.242 times that of the earth from the sun.

5. The mean distance of mars from sun is 1.524 times the distance of the earth from sun. Compute the period of revolution of mars around the sun.

Solution: we know that $T^2 \propto a^3$

$$\frac{T_1^2}{T_2^2} = \frac{a_1^3}{a_2^3}$$

$$period\ of\ revolution\ of\ earth\ T_2 = 1\ year$$

$$Mars\ \ period\ of\ revolution\ T_1 = ?$$

$radius\ of\ the\ earth\ a_2 = a$,

$radius\ of\ the\ Mars\ \ a_1 = 1.524$ a

$$T_1 = T_2 (\frac{a_1}{a_2})^{3/2}$$

$$T_1 = 1\ x\ (1.524)^{3/2} = 1.88$$

Mars revolves round the sun is 1.88 earth years.