VIBRATING STRINGS

TRANSVERSE WAVE IN A STRING

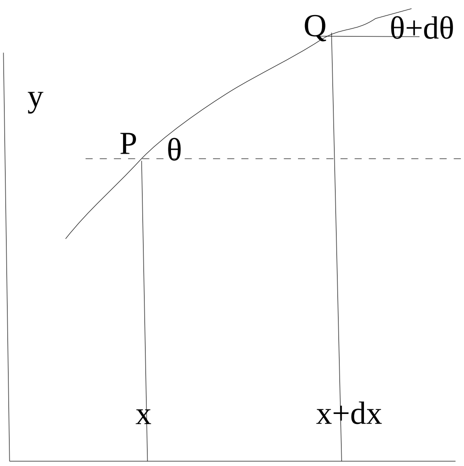
Consider a string of length 'l' which is fixed at both ends with tension 'T'. If we pluck the string then a disturbance will be created in it. This disturbance will be propagated from left to right in the form of wave. Because of this disturbance the particles start making oscillations perpendicular to the direction of propagation of the wave. This creates transverse wave in string.

The speed of a wave depends on the properties of the medium. It is related to wavelength and frequency of the wave. If a wave travels through a medium then it causes the particles in that medium to oscillate. For this to happen the medium need to possess mass and elasticity. These properties of the medium determine the velocity of wave in that medium.

EQUATION OF MOTION

Consider a string of length 'l', uniform cross section α, linear density μ which is fixed between two rigid supports. Let us consider a small segment dx in the string and a symmetrical pulse is travelling from left to right along the string with velocity v. Force equal to the magnitude of tension pulls tangentially the sement at P and Q.

Horizontal components of forces will cancel each other and the net force is given by sum of all vertical components.



$$F_y = (T\sin\theta)_{x+dx} - (T\sin\theta)_x \quad ...(1)$$

Using Taylor series expansion

$$(T\sin\theta)_{x+dx} = T(\sin\theta) + T\frac{\partial}{\partial x}(\sin\theta)dx + T\frac{\partial^2}{\partial x^2}(\sin\theta)\frac{(dx)^2}{2} + ...$$

When θ is small $\sin\theta = \frac{\partial y}{\partial x}$ and higher terms are neglected

$$(T\sin\theta)_{x+dx} = T\sin\theta + T\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)dx$$

substituting in eqn (1) gives $F_y = T\sin\theta + T\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)dx - (T\sin\theta)$

$$F_y = T\frac{\partial^2 y}{\partial x^2}dx \quad ...(2)$$

If μ is linear density of the string then mass of the element dx will be μ dx and acceleration will be

$\frac{\partial^2 y}{\partial t^2}$. From Newton's II law force is $F_y = \mu\, dx\frac{\partial^2 y}{\partial t^2}$

Balancing eqn (2) with Newton's II law $\mu dx \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2} dx$

$$\Rightarrow \frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu} \frac{\partial^2 y}{\partial x^2} \quad ...(3)$$

Comparing the above eqn with general wave eqn $\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$

From the above two eqns, Velocity of the wave is $v = \sqrt{\left(\frac{T}{\mu}\right)}$

The above second order differential eqn represents eqn of transverse waves travelling in a string with velocity v.

It is observed that the velocity of a wave in a stretched string is directly proportional to square root of tension acting on the string and inversely proportional to square root of linear density of the string; it is independent of frequency of the wave.

SOLUTION AND ITS PHYSICAL SIGNIFICANCE:

Solution to eqn (3) depends on variables 'x' and 't' and will be represented by using periodic functions.

Different forms of solutions to eqn (3) are as follows

$y(x,t) = A \sin(\omega t \pm kx)$ or $y(x,t) = A \cos(\omega t \pm kx)$ or $y(x,t) = A e^{i(\omega t \pm kx)}$

Solutions with positive and negative terms represent waves travelling in forward and backward directions (negative and positive x directions).

1. y is transverse displacement of any string element; it is a function of time t and position of the element x.
2. A is amplitude of the wave which is always a positive quantity.
3. ω is angular frequency. $\omega = 2\pi f$ f is frequency of vibration
4. k is propagation constant or angular wave number. $k = \frac{2\pi}{\lambda}$ λ is wavelength of the wave.

ω and k are related by eqn $\frac{\omega}{k} = v$ velocity of the wave.

$$\Rightarrow v = \frac{2\pi f}{\left(\frac{2\pi}{\lambda}\right)} \qquad \Rightarrow v = f\lambda$$

This eqn gives the relation of velocity with frequency and wavelength of the wave.

MODES OF VIBRATIONS OF A STRETCHED STRING CLAMPED AT BOTH ENDS

Consider a string of length l, linear density μ which is fixed at both ends. When the string is stretched through tension T transverse waves will propagate in it. The wave travelling in one direction reflects at the end and returns inverted, forming nodes at the ends. This forms stationary waves in the string and the string is said to resonate at resonant frequencies.

Displacements of the waves are given as $y_1 = A_1 e^{i(\omega t - kx)}$ and $y_2 = A_2 e^{i(\omega t + kx)}$

From superposition principle, resultant displacement is $y = y_1 + y_2$

$$\Rightarrow y = A_1 e^{i(\omega t - kx)} + A_2 e^{i(\omega t + kx)} \quad ...(1)$$

Since string is fixed at both ends, boundary conditions are

I $y=0$ at $x=0$ and

II $y=0$ at $x=l$

From condition I eqn (1) becomes $0 = A_1 e^{i\omega t} + A_2 e^{i\omega t}$ $\Rightarrow (A_1 + A_2) e^{i\omega t} = 0$

as $e^{i\omega t} \neq 0$ $\Rightarrow (A_1 + A_2) = 0$ $\Rightarrow A_2 = -A_1$

Substituting in eqn (1) gives $\Rightarrow y = A_1 e^{i(\omega t - kx)} - A_1 e^{i(\omega t + kx)}$

$$\Rightarrow y = A_1 e^{i\omega t} \left(e^{-ikx} - e^{ikx}\right)$$

$\because$ $\frac{e^{ikx} - e^{-ikx}}{2i} = \sin(kx)$ $\Rightarrow y = -2i A_1 e^{i\omega t} \sin kx$ ...(2)

Applying condition II, eqn (2) becomes $\Rightarrow 0 = -2i A_1 e^{i\omega t} \sin kl$

$\because$ $A_1 \neq 0$ and $e^{i\omega t} \neq 0$ $\Rightarrow \sin kl = 0$ $\Rightarrow k_n l = n\pi$ or $k_n = \frac{n\pi}{l}$

we know that angular frequency $\omega_n = k_n v$ $\Rightarrow \omega_n = \frac{n\pi}{l} v$

Frequency $f = \frac{\omega}{2\pi}$ $\Rightarrow f_n = \frac{1}{2\pi} \frac{n\pi}{l} v$ $\Rightarrow f_n = \frac{n}{2l} v$

$\because$ $v = \sqrt{\left(\frac{T}{\mu}\right)}$ $\Rightarrow f_n = \frac{n}{2l} \sqrt{\left(\frac{T}{\mu}\right)}$

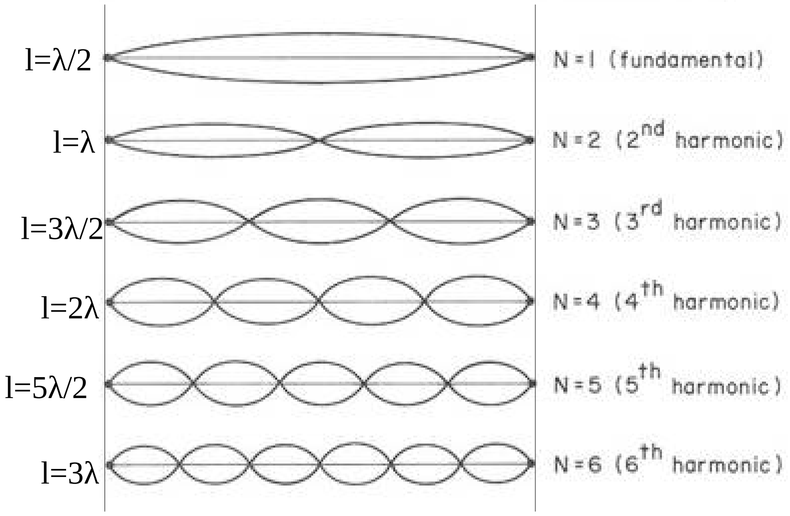
The above eqn indicates that a string fixed at both ends vibrates with only certain frequencies. Such frequencies are known as harmonics. n is known as harmonic number.

Note: General expression for displacement of a transverse wave along a stretched string fixed at both ends is given by

$$y_n=\sum_i -2\,i\,A_n e^{i\omega_n t}\sin k_n l$$

OVERTONES OR HARMONICS:

Consider a string of length l, linear density μ which is fixed at both ends. When the string is stretched through tension T transverse waves will propagate in it.



The expression for frequency of wave in a stretched string ,fixed at both ends is $f_n=\frac{n}{2l}\sqrt{(\frac{T}{\mu})}$

For n=1, the frequency will be $f_1=\frac{1}{2l}\sqrt{(\frac{T}{\mu})}$

This is the lowest possible frequency known as fundamental frequency or first harmonic.

All higher frequencies are integer multiples of fundamental frequency.

For n=2 $f_2=\frac{2}{2l}\sqrt{(\frac{T}{\mu})}=2f_1$, known as second harmonic or first overtone.

For n=3 $f_3=\frac{3}{2l}\sqrt{(\frac{T}{\mu})}=3f_1$, known as third harmonic or second overtone ...

$$f_1:f_2:f_3...=1:2:3...$$

Overtones or harmonics represent resonant frequencies. The term overtone is generally applied to high frequency standing waves. The term harmonic is used in cases where overtones are integral multiples of fundamental frequency.

Note: Overtones play an important role in the analysis of quality of musical tones in mucical instruments.

LAWS OF VIBRATIONS (MERSENNE'S LAWS):

Mersenne's laws desribe the dependence of fundamental frequency of vibration of a stretched string on its length (l), Tension(T) and linear density(μ).

1. The fundamental frequency of vibration is inversely proportional to the vibrating length, for a string of constant tension and linear density.

$$f \propto \frac{1}{l} \quad \text{(T and μ are constants)}$$

2. The fundamental frequency of vibration is directly proportional to the squareroot of tension acted on a string, whose length and linear density are constants.

$$f \propto \sqrt{T} \quad \text{(l and μ are constants)}$$

3. The fundamental frequency of vibration is inversely proportional to the squareroot of linear density of the string with constant vibrating length and tension.

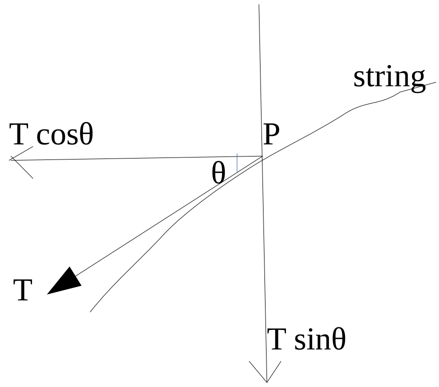
$$f \propto \frac{1}{\sqrt{\mu}} \quad \text{(l and T are constants)}$$

Note: The laws were first proposed by French mathematician and music theorist Marin Marsenne. Mersenne's laws play an important role in construction and operation of string instruments such as violin, guitar, piano etc.
Total tension must be adjusted in a manner such that they should maintain proper pitch.

ENERGY TRANSPORT:

Let us consider a string of length 'l' and linear density 'μ'. By continuously oscillating one end of the string we can generate a wave in it. It means we are providing energy continuously for the motion and stretching of the string. As the wave moves into sections that are at rest previously, energy transports into those new sections of the string.

Consider a point P in the stretched string. Force acting at P can be resolved into horizontal and vertical components. But, force acting is equal and opposite to the vertical component of tension in the string (Since vertical component brings the system back to equilibrium position).
Hence $F_y = -T \sin\theta$



For small values of θ, $\sin\theta \simeq \frac{\partial y}{\partial x} \quad \therefore \quad F_y = -T\frac{\partial y}{\partial x}$

Also, expression for velocity is $u = \frac{\partial y}{\partial t}$

Power transmitted by wave is P= force x velocity

$$\Rightarrow P = -T\frac{\partial y}{\partial x}\frac{\partial y}{\partial t} \quad ...(1)$$

$\because \quad y = A\sin(\omega t - kx)$ is expression for displacement of wave in a stretced string

$\Rightarrow \frac{\partial y}{\partial x} = -Ak\cos(\omega t - kx)$ and $\frac{\partial y}{\partial t} = A\omega\cos(\omega t - kx)$

substituting in eqn (1) gives $P = T A^2 k\omega\cos^2(\omega t - kx)$

$\because \quad k = \frac{\omega}{v}$ $\quad \Rightarrow P = T A^2 \frac{\omega^2}{v}\cos^2(\omega t - kx)$

$\because \quad \omega = 2\pi f$ $\quad \Rightarrow P = T A^2 \frac{4\pi^2 f^2}{v}\cos^2(\omega t - kx)$

Expression for average power is $P_{avg} = T A^2 \frac{4\pi^2 f^2}{v}\frac{1}{2}$ ($\because$ avg value of $\cos^2\theta = 1/2$)

$\because \quad v = \sqrt{(\frac{T}{\mu})}$ $\quad \Rightarrow P_{avg} = v^2 \mu A^2 \frac{2\pi^2 f^2}{v}$

$$\Rightarrow P_{avg} = 2\pi^2 f^2 A^2 v\mu$$

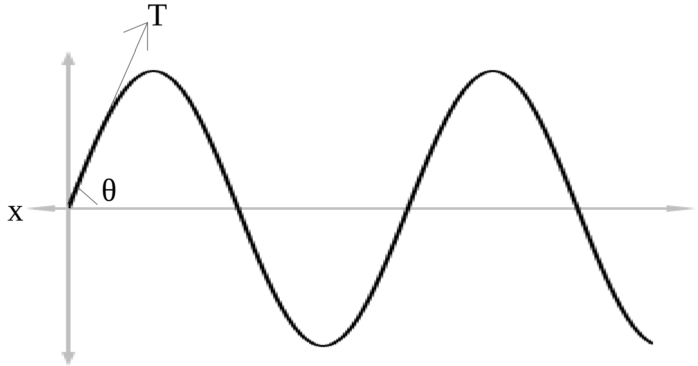
From the above eqn, it is clear that average power of a wave is

1. directly proportional to square of frequency
2. directly proportional to square of amplitude
3. directly proportional to wave velocity
4. directly proportional to linear density

IMPEDANCE TO WAVE MOTION:

Inherent properties of medium which opposes the propagation of wave is known as impedance. Because of this impedance of the medium, the loss in energy occurs. Impedance is a complex quantity.

The ratio of external force (F) to the velocity (u) of the particle tells the inertia offered by the medium, which is defined as impedance (z). $z = \frac{F}{u}$



Consider a string of length l, linear density μ, fixed at both ends due to tension T.

External force applied is $F = F_0 e^{i\omega t}$ ...(1)

Transverse component of tension applied is T sinθ $\therefore$ $F = -T \sin\theta$

For small values of θ $\sin\theta \simeq \frac{\partial y}{\partial x}$ $\therefore$ $F = -T \frac{\partial y}{\partial x}$ …(2)

From eqns (1) and (2) $F_0 e^{i\omega t} = -T \frac{\partial y}{\partial x}$ ...(3)

Displacement of a transverse wave in a stretched string is $y = A e^{i(\omega t - kx)}$ ...(4)

Differentiate w.r.t x $\frac{\partial y}{\partial x} = -i k A e^{i(\omega t - kx)}$

$$\Rightarrow \left(\frac{\partial y}{\partial x}\right)_{x=0} = -i k A e^{i\omega t}$$

substituting in eqn (3) gives $F_0 e^{i\omega t} = i T k A e^{i\omega t}$

$\because$ $k = \frac{\omega}{v}$ $\Rightarrow F_0 = i T \frac{\omega}{v} A$

Amplitude of vibration is $A = \frac{F_0 v}{i T \omega}$

substituting the expression for A in eqn (4) gives $y = \frac{F_0 v}{i T \omega} e^{i(\omega t - kx)}$

velocity is $u = \frac{\partial y}{\partial t} = \frac{F_0 v}{i T \omega} i \omega e^{i(\omega t - kx)}$

$$\Rightarrow u = \frac{F_0 v}{T} e^{i(\omega t - kx)} \qquad \Rightarrow u_{x=0} = \frac{F_0 v}{T} e^{i\omega t}$$

from the definition of impedance $z = \frac{F}{u}$

substituting the values of F and u gives $z = \frac{F_0 e^{i\omega t}}{\frac{F_0 v}{T} e^{i\omega t}}$ $\Rightarrow z = \frac{T}{v}$

$\because$ $v = \sqrt{\left(\frac{T}{\mu}\right)}$ $\Rightarrow T = \mu v^2$

substituting in the above expression for impedance gives $z=\frac{\mu v^2}{v}$ $\Rightarrow z=\mu v$ or $\Rightarrow z=\sqrt{T\mu}$

∴ Impedance for mechanical waves is the product of wave velocity and linear density of the string.

or

Impedance for transverse waves in a string is squareroot of product of tension and linear density of the string.

PROBLEMS

1. A wave travelling along a string is described by $y(x,t)=0.00327\sin(72.1x-2.72t)$ in SI units.
what is the amplitude, wavelength, time period, frequency of the wave?
What is the displacement at x=22.5 cm and t=18.9 s?

2. A string of linear density 525 gm/m is under tension 45 N. A wave of frequency 120 Hz and amplitude 8.5 mm is passed through the string. Find the rate at which energy transport?

3. A string of length 2 m, mass 0.6 kg is under tension 500 N. Find the velocity of wave in that string?

4. A string of length 1 m, radius 1 mm, density 7930 kg/m^3 is under tension 10^4 N. Determine the values of fundamental frequency, first overtone and second overtone and the corresponding wavelengths.

5. Tension applied to a string is 1000 N/m. Velocity of transverse wave in the string is 200 m/s. Find the impedance offered by the medium.