VIBRATIONS OF BARS

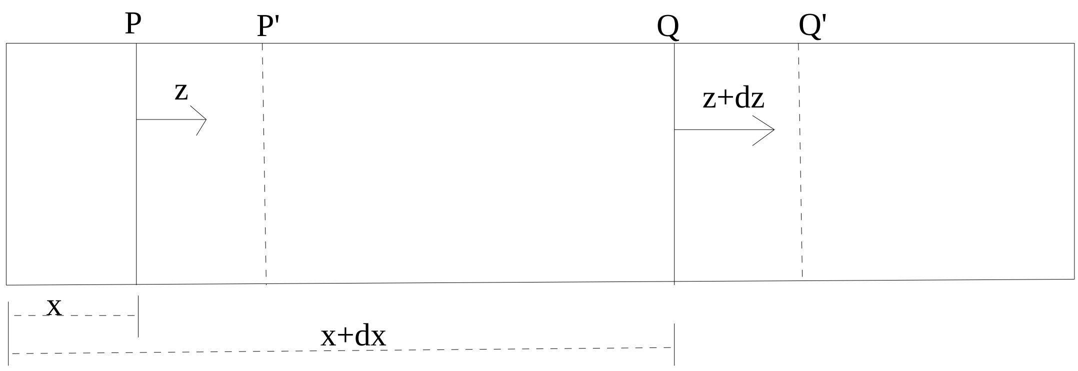
Bars are components which have one dimension (length) larger than the other two dimensions. In general, when bars are excited both longitudinal and transverse waves are generated. This is because of the coupling between stiffness and tension of the bar.

During longitudinal wave propagation, disturbance propagates along the axis of the bar. Because of the longitudinal vibrations there will be an increase in length and decrease in lateral magnitude of the bar. When the diameter of the bar reduces transverse wave propagation will be negligible.

LONGITUDINAL WAVES IN A BAR:

consider a uniform elastic bar of length 'l' and cross section 'α'. Area is perpendicular to main axis of the bar(x-axis). The material properties of bar are its density 'ρ' and Young's modulus 'Y', which are constant at any given cross section. By applying a longitudinal force 'F', the bar is allowed to make oscillations. The elastic displacements are described as z(x,t) due to the applied force.

Consider a small segment PQ. Coordinates at P and Q are x and x+dx respectively.
Because of the applied force, P is displaced to P' and Q is displaced to Q'. The corresponding displacements of P and Q are z and z+dz along positive z direction.



Newton's II law gives the force acting on the bar as $F=\rho\,\alpha\,dx\,\frac{\partial^2 z}{\partial t^2}$ ...(1)

Displacement can be expressed as $z+dz=z+\frac{\partial z}{\partial x}dx$ (From Taylor series expansion, neglecting higher order terms)

Along the length of the bar, extension of the element PQ is $(z+dz)-z$

$$= z+\frac{\partial z}{\partial x}dx-z = \frac{\partial z}{\partial x}dx$$

Longitudinal strain is defined as ratio of change in length to the original length of the bar.

Longitudinal strain is $\frac{\frac{\partial z}{\partial x}dx}{dx} = \frac{\partial z}{\partial x}$

From the definition of elastic modulus $Y = \frac{longitudinal\, stress}{longitudinal\, strain}$

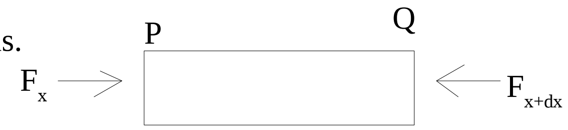
$$\Rightarrow longitudinal\, stress = longitudinal\, strain(Y) \quad = \quad Y\frac{\partial z}{\partial x}$$

$$longitudinal\, stress = \frac{Force}{Area\, of\, cross\, section}$$

$$\Rightarrow Force = (longitudinal\, stress)(Area\, of\, cross\, section) \quad = \quad \Rightarrow F = \alpha Y\frac{\partial z}{\partial x} \quad ...(2)$$

The above eqn gives axial force at a cross section of the bar.

But the force varies under dynamic conditions.



F_x is force acting at point P and F_{x+dx} is force acting at point Q.

Net force acting on segment PQ is $F = F_{x+dx} - F_x$

$$\Rightarrow F = F_x + \frac{\partial F_x}{\partial x}dx - F_x \quad = \quad \frac{\partial F_x}{\partial x}dx$$

Substituting the expression for force from eqn (2) gives $F = Y\alpha\frac{\partial}{\partial x}(\frac{\partial z}{\partial x})dx$

$$\Rightarrow F = Y\alpha\frac{\partial^2 z}{\partial x^2}dx \quad ...(3)$$

From eqns. (1) and (3) $\rho\alpha\, dx\frac{\partial^2 z}{\partial t^2} = Y\alpha\frac{\partial^2 z}{\partial x^2}dx$

$$\frac{\partial^2 z}{\partial t^2} = \frac{Y}{\rho}\frac{\partial^2 z}{\partial x^2} \quad ...(4)$$

Above eqn represents longitudinal wave eqn in a bar. It is a second order differential eqn with longitudinal displacement 'z'.

General wave eqn. is represented as $\frac{\partial^2 z}{\partial t^2} = c^2\frac{\partial^2 z}{\partial x^2}$

Comparing the above eqns gives the expression for velocity of longitudinal wave in a bar

$$c = \sqrt{\frac{Y}{\rho}}$$

Velocity of a longitudinal wave in a bar is

1. directly proportional to squareroot of Young's modulus of the material of bar.
2. inversely proportional to squareroot of density of the material of bar.

SOLUTION:

Solution to eqn of motion of a longitudinal wave in a bar will be of the form

$$z=f(ct-x)+f(ct+x)$$

c is velocity of longitudinal waves in bar $c=\sqrt{\frac{Y}{\rho}}$

solution to eqn (4) can be of the form $z=Ae^{i(\omega t-kx)}+Be^{i(\omega t+kx)}$ ...(5)

1. z is longitudinal displacement of the bar, is a function of time 't' and position 'x';
2. A and B are constants;
3. ω is angular frequency. $\omega=2\pi f$ f is frequency of vibration.
4. k is propagation constant $k=\frac{\omega}{c}=\frac{2\pi}{\lambda}$. λ is wavelength of the wave.
5. ω and k are related by eqn $\frac{\omega}{k}=c$ velocity of the wave.

BOUNDARY CONDITIONS:

1. At a fixed point of the bar, displacement is zero. $z=0$
2. At a free end of the bar, force acting at that point is zero. $\frac{\partial z}{\partial x}=0$

LONGITUDINAL VIBRATIONS IN A BAR FIXED AT BOTH ENDS:

Consider a bar of length 'l', mass 'm', density 'ρ' and uniform cross section α. It is fixed at both the ends i.e., at x=0 and at x=l and is allowed to make longitudinal vibrations. For a bar of length 'l' fixed at both ends, displacement will be zero at the end points.

Boundary conditions are as follows

1. Displacement $z=0$ at $x=0$
2. displacement $z=0$ at $x=l$

Let us apply the boundary conditions to general solution to longitudinal waves in a bar

Displacement of the wave is $z=Ae^{i(\omega t-kx)}+Be^{i(\omega t+kx)}$ ...(1)

From condition (1) $Ae^{i\omega t}+Be^{i\omega t}=0$

$$\Rightarrow(A+B)e^{i\omega t}=0 \quad \Rightarrow(A+B)=0 \quad (\because \; e^{i\omega t}\neq 0 \;) \quad \therefore \; B=-A$$

Substitute the above relation in eqn (1) $z=A\,e^{i(\omega t-kx)}-A\,e^{i(\omega t+kx)}$

$$\Rightarrow z=A\,e^{i\omega t}(e^{-ikx}-e^{ikx})$$

$$\Rightarrow z = -2\,i\,A e^{i\omega t}\sin kx$$

Applying condition (2) gives $-2\,i\,A\,e^{i\omega t}\sin kl = 0$

$$\Rightarrow \sin kl = 0 \quad \Rightarrow kl = 0, \pi, 2\pi, \ldots n\pi \quad \Rightarrow k_n l = n\pi \quad \Rightarrow k_n = \frac{n\pi}{l}$$

$$\because \quad k_n = \frac{\omega_n}{c} \quad \Rightarrow \frac{\omega_n}{c} = \frac{n\pi}{l} \quad \Rightarrow \omega_n = \frac{n\pi c}{l}$$

$$\because \quad \omega_n = 2\pi f_n \quad \Rightarrow f_n = \frac{\omega_n}{2\pi} \quad \Rightarrow f_n = \frac{1}{2\pi}\frac{n\pi c}{l} \quad \Rightarrow f_n = \frac{nc}{2l} \quad \text{or} \quad \Rightarrow f_n = \frac{n}{2l}\sqrt{\frac{Y}{\rho}}$$

For $n = 1$ $f_1 = \frac{c}{2l}$ known as fundamental frequency

$n = 2$ $f_2 = \frac{2c}{2l} = 2f_1$ known as first overtone

$n = 3$ $f_2 = \frac{3c}{2l} = 3f_1$ known as second overtone ...

For a bar fixed at both ends, frequencies of overtones are integral mutiples of fundamental frequency.

LONGITUDINAL WAVES OF A BAR FREE AT BOTH ENDS:

Consider a bar of length 'l' , mass 'm', density 'ρ' and uniform cross section α. It is free at both the ends i.e., at x=0 and at x=l and is allowed to make longitudinal vibrations. When bar is free at both ends, displacement at the end points will be maximum.

Boundary conditions are

1. At $x = 0$ $\frac{\partial z}{\partial x} = 0$
2. At $x = l$ $\frac{\partial z}{\partial x} = 0$

Let us apply the boundary conditions to general solution of longitudinal waves in a bar

Displacement of the wave is $z = Ae^{i(\omega t - kx)} + Be^{i(\omega t + kx)}$ ...(1)

diff. w.r.t 'x' $\Rightarrow \frac{\partial z}{\partial x} = -Ak e^{i(\omega t - kx)} + Bk e^{i(\omega t + kx)}$ ...(2)

Applying condition (1) gives $\frac{\partial z}{\partial x} = -Ak e^{i\omega t} + Bk e^{i\omega t} = 0$

$$\Rightarrow(-A+B)k\,e^{i\omega t}=0 \qquad \Rightarrow A=B$$

Substitute above relation in eqn (1) $\quad z=Ae^{i(\omega t-kx)}+Ae^{i(\omega t+kx)} \qquad \Rightarrow z=Ae^{i\omega t}(e^{-ikx}+e^{ikx})$

$$\Rightarrow z=2\,A\,e^{i\omega t}\cos kx$$

diff. above eqn w.r.t 'x' $\quad \frac{\partial z}{\partial x}=-2k\,A\,e^{i\omega t}\sin kx$

Applying condition (2) in above eqn gives $\quad -2k\,A\,e^{i\omega t}\sin kl=0$

$$\Rightarrow \sin kl=0 \qquad \Rightarrow kl=0,\pi,2\pi,\ldots n\pi \qquad \Rightarrow k_n l=n\pi \qquad \Rightarrow k_n=\frac{n\pi}{l}$$

$$\because \quad k_n=\frac{\omega_n}{c} \qquad \Rightarrow \frac{\omega_n}{c}=\frac{n\pi}{l} \qquad \Rightarrow \omega_n=\frac{n\pi c}{l}$$

$$\because \quad \omega_n=2\pi f_n \qquad \Rightarrow f_n=\frac{\omega_n}{2\pi} \qquad \Rightarrow f_n=\frac{1}{2\pi}\frac{n\pi c}{l} \qquad \Rightarrow f_n=\frac{nc}{2l}$$

For $\quad n=1 \quad f_1=\frac{c}{2l}$ known as fundamental frequency

$n=2 \quad f_2=\frac{2c}{2l}=2f_1$ known as first overtone

$n=3 \quad f_2=\frac{3c}{2l}=3f_1$ known as second overtone ...

The above relations show that overtones are integral multiples of fundamental frequency. This is similar to a bar fixed at both ends.

LONGITUDINAL VIBRATIONS OF A BAR FIXED AT ONE END AND FREE AT THE OTHER END:

Consider a bar of length 'l' , mass 'm', density 'ρ' and uniform cross section α. It is fixed at x=0 and free at x=l and is allowed to make longitudinal vibrations.

Boundary conditions are

1. At fixed end i.e., at $x=0 \quad z=0$
2. At free end i.e., at $x=l \quad \frac{\partial z}{\partial x}=0$

Let us apply the boundary conditions to general solution of longitudinal waves in a bar

Displacement of the wave is $\quad z=Ae^{i(\omega t-kx)}+Be^{i(\omega t+kx)}$ ...(1)

Applying condition (1) gives $Ae^{i\omega t}+Be^{i\omega t}=0$

$$\Rightarrow (A+B)e^{i\omega t}=0 \quad \Rightarrow A+B=0 \quad (\because \; e^{i\omega t}\neq 0 \;)$$

$$\therefore \; B=-A$$

Substitute the above relation in eqn (1) $z=A\,e^{i(\omega t-kx)}-A\,e^{i(\omega t+kx)}$ $\Rightarrow z=A\,e^{i\omega t}\left(e^{-ikx}-e^{ikx}\right)$

$$\Rightarrow z=-2\,i\,A\,e^{i\omega t}\sin kx$$

diff. above eqn w.r.t x $\Rightarrow \frac{dz}{dx}=-2ik\,A\,e^{i\omega t}\cos kx$

Applying condition (2) $-2\,ik\,A\,e^{i\omega t}\cos kl=0$ $\Rightarrow \cos kl=0$

$\therefore \quad kl=\frac{\pi}{2},3\frac{\pi}{2},5\frac{\pi}{2},\ldots(2n-1)\frac{\pi}{2}$ or $k_n=(2n-1)\frac{\pi}{2l}$

$\because \quad k_n=\frac{\omega_n}{c} \quad \Rightarrow \frac{\omega_n}{c}=(2n-1)\frac{\pi}{2l} \quad \Rightarrow \omega_n=(2n-1)\frac{\pi c}{2l}$

$\because \quad \omega=2\pi f_n \quad \Rightarrow f_n=\frac{\omega_n}{2\pi} \quad \Rightarrow f_n=\frac{1}{2\pi}\frac{(2n-1)\pi c}{2l} \quad \Rightarrow f_n=(2n-1)\frac{c}{4l}$

For $n=1$ $f_1=\frac{c}{4l}$ known as fundamental frequency

$n=2$ $f_2=\frac{3c}{4l}=3f_1$ known as first overtone

$n=3$ $f_2=\frac{5c}{4l}=5f_1$ known as second overtone ...

Frequency of overtones produced in a free-free bar are odd integral multiples of fundamental frequency.

LONGITUDINAL VIBRATIONS IN A BAR FIXED AT THE MID POINT:

Consider a bar of length 'l' , mass 'm', density 'ρ' and uniform cross section α. It is fixed at mid point i.e., at $x=\frac{l}{2}$ and is allowed to make longitudinal vibrations.

Boundary conditions are

1. At free end i.e., at $x=0$ $\frac{\partial z}{\partial x}=0$

2. At fixed end i.e., at $x=\frac{l}{2}$ $z=0$

Let us apply the boundary conditions to general solution of longitudinal waves in a bar

Displacement of the wave is $z = Ae^{i(\omega t - kx)} + Be^{i(\omega t + kx)}$ ...(1)

diff. w.r.t 'x' $\Rightarrow \frac{\partial z}{\partial x} = -A k e^{i(\omega t - kx)} + B k e^{i(\omega t + kx)}$ ...(2)

Applying condition (1) gives $-A k e^{i\omega t} + B k e^{i\omega t} = 0 \quad \Rightarrow k e^{i\omega t}(-A + B) = 0 \quad \Rightarrow (-A + B) = 0$

$$\Rightarrow A = B$$

Substitute the above relation in eqn (1) $z = Ae^{i(\omega t - kx)} + Ae^{i(\omega t + kx)} \quad \Rightarrow z = Ae^{i\omega t}(e^{-ikx} + e^{ikx})$

$$\Rightarrow z = 2Ae^{i\omega t}\cos kx$$

Applying condition (2) gives $2Ae^{i\omega t}\cos\frac{kl}{2} = 0 \quad \Rightarrow \cos\frac{kl}{2} = 0$

$\therefore \quad \frac{kl}{2} = \frac{\pi}{2}, 3\frac{\pi}{2}, 5\frac{\pi}{2}, \ldots (2n-1)\frac{\pi}{2} \qquad \frac{k_n l}{2} = (2n-1)\frac{\pi}{2}$ or $k_n = (2n-1)\frac{\pi}{l}$

$\because \quad k_n = \frac{\omega_n}{c} \quad \Rightarrow \frac{\omega_n}{c} = (2n-1)\frac{\pi}{l} \quad \Rightarrow \omega_n = (2n-1)\frac{\pi c}{l}$

$\because \quad \omega_n = 2\pi f_n \quad \Rightarrow f_n = \frac{\omega_n}{2\pi} \quad \Rightarrow f_n = \frac{1}{2\pi}\frac{(2n-1)\pi c}{l} \quad \Rightarrow f_n = (2n-1)\frac{c}{2l}$

For $n = 1 \quad f_1 = \frac{c}{2l}$ known as fundamental frequency

$n = 2 \quad f_2 = \frac{3c}{2l} = 3f_1$ known as first overtone

$n = 3 \quad f_2 = \frac{5c}{2l} = 5f_1$ known as second overtone ...

For a bar fixed at mid point, overtones are odd integral multiples of fundamental frequency, as in the case of a fixed-free bar.

PROBLEMS

1. In steel velocity of sound is 5050 m/sec. If density of steel is 7700 kg/m^3 then determine Young's modulus of steel.
2. 1m length of a bar is fixed at centre. Calculate fundamental frequency and first overtone of longitudinal wave.
3. In copper whose density is 8890 kg/m^3 velocity of sound is 3560 m/sec . Find Young's modulus of copper.